Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 25, 2002*
Accepted: *June 17, 2002*

Superstrings on pp-wave backgrounds and symmetric orbifolds

Yasuaki Hikida and Yuji Sugawara

Department of Physics, Faculty of Science, University of Tokyo
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan
E-mail: hikida@hep-th.phys.s.u-tokyo.ac.jp, sugawara@hep-th.phys.s.u-tokyo.ac.jp

Abstract: We study the superstring theory on pp-wave background with NSNS-flux that is realized as the Penrose limit of $AdS_3 \times S^3 \times M^4$, where M^4 is T^4 or $T^4/\mathbb{Z}_2$ ($\cong K3$). Quantizing this system in the covariant gauge, we explicitly construct the space-time supersymmetry algebra and the complete set of DDF operators. We analyse the spectrum of physical states by using the spectrally flowed representations of current algebra. This spectrum is classified by the "short string sectors" and the "long string sectors" as in AdS_3 string theory. The states of the latter propagate freely along the transverse plane of pp-wave background, but the states of the former do not. We compare the short string spectrum with the BPS and almost BPS states which have large R-charges in the symmetric orbifold conformal theory, which is known as the candidate of dual theory of superstrings on $AdS_3 \times S^3 \times M^4$. We show that every short string states can be embedded successfully in the single particle Hilbert space of symmetric orbifold conformal theory.

Keywords: Superstrings and Heterotic Strings, Conformal Field Models in String Theory, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Recently the string theories on pp-wave backgrounds have been studied intensively. They are realized by the Penrose limits of the string theories on AdS spaces [1] and it has just discovered that the string theory on pp-wave background with RR-flux can be quantized in the light-cone gauge [2]. The authors of [3] applied these facts to the AdS/CFT correspondence [4] and they have made a remarkable progress beyond the supergravity approximation. More precisely, they considered the "almost BPS states" with large R-charges, which slightly break supersymmetry, and succeeded in constructing the single trace operators describing the string excitations in the large-N supersymmetric Yang-Mills theory. Many developments have been presented since then. The orbifoldizations [5] and the extensions including D-branes [6] have been investigated widely and the other recent developments are given in [7]–[13].

JHEP06(2002)037

The cases of the Penrose limits of $AdS_3 \times S^3$ are somewhat special because we can also consider the NSNS-flux. Such backgrounds are exactly soluble as noncompact WZW models associated with the 6-dimensional Heisenberg group (H_6), which are simple generalizations of the Nappi-Witten models [14]. Contrary to the cases with RR-flux,[1] we can quantize these systems covariantly by using the standard NSR formalism of superstring theory.

The superstring theory on $AdS_3 \times S^3 \times M^4$ ($M^4 = T^4$ or $K3$) with NSNS-flux is realized as the near horizon limit of NS1-NS5 system [15]. This is a system of great importance because it gives a possibility to make analysis beyond the supergravity approximation. The famous candidate of the dual theory is the 2-dimensional $\mathcal{N} = (4,4)$ non-linear σ-model on the symmetric orbifold $\mathrm{Sym}^{Q_1 Q_5}(M^4) \equiv (M^4)^{Q_1 Q_5}/S_{Q_1 Q_5}$ [16, 17]. (Q_1 and Q_5 are the NS1 and NS5 charges, respectively.) There are many attempts to understand this duality from the world-sheet picture of string theory (e.g., [15, 18, 19]). One of the important tests of this duality is the comparison of the spectrum of BPS states. The BPS states with small R-charges are explicitly identified [20, 21], and the higher R-charged BPS states are also reproduced successfully in [22, 23] with the help of the spectral flow symmetry [24, 25]. However, the analysis beyond the BPS spectrum has been difficult and still incomplete.

In order to overcome this difficulty we shall follow the idea of [3]. More precisely, we will concentrate on the states which possess very large R-charges and break the BPS condition slightly. These states are often called as "almost BPS states". The sectors of such states are well described by taking the Penrose limit. Under this limit, the string theory on $AdS_3 \times S^3 \times M^4$ reduces to the WZW model on $H_6 \times M^4$, as we mentioned above. This model is much easier to treat and the string Hilbert space is expected to be completely analysed.

In this paper, motivated by these observations, we study the superstrings on the pp-wave backgrounds with NSNS-flux, more precisely, $H_6 \times M^4$ spaces ($M^4 = T^4$, $T^4/\mathbb{Z}_2$) in the covariant gauge quantization. We construct the space-time supersymmetry algebra by using the physical string vertices in the manner similar to [15] and present the complete set of DDF operators. From this result we investigate the spectrum of physical states. In this analysis the spectrally flowed representation plays an essential role just as in the case of AdS$_3$ string. The spectrum is classified by the "short string sectors" and the "long string sectors" as in AdS$_3$ string theory [25]. The strings of former sectors cannot propagate along the transverse plane of pp-wave geometry, while the strings of latter sectors freely propagate and have a continuous spectrum of light-cone energies. Finally we compare the short string spectrum with the (almost) BPS states in the conformal field theory on symmetric orbifold. We find out successfully the natural embedding of all the string states in the single particle Hilbert space of symmetric orbifold theory. We also comment on the existence of many missing states, which should be identified with the non-perturbative excitations in string theory side.

This paper is organized as follows. In section 2, which is a preliminary section, the space-time supersymmetry ("super pp-wave algebra") in the $H_6 \times T^4$ background is examined by contracting the supersymmetry algebra of superstring theory on $AdS_3 \times S^3 \times T^4$.

[1]The quantization of the case with RR-flux in the covariant gauge is discussed recently in [10].

In section 3, we quantize the superstring theory on $H_6 \times T^4$ in the covariant gauge and construct the physical vertices generating the super pp-wave algebra and the complete set of DDF operators. From this result we analyse the spectrum of physical states explicitly. In section 4, starting with a short review of the symmetric orbifold, we compare the (almost) BPS spectrum with the string spectrum on the pp-wave background. We also discuss the extension to the case of $H_6 \times T^4/\mathbb{Z}_2 (\cong H_6 \times K3)$ in section 5, and the section 6 is devoted to conclusion and discussion.

2. Space-time superalgebra of pp-wave background

In this preliminary section we examine the supersymmetric algebra of pp-wave background by contracting that of $AdS_3 \times S^3$. It is well-known that the supersymmetry in the background $AdS_3 \times S^3$ is represented by the super Lie group $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$. The even part corresponds to the isometry of this background. The isometry of AdS_3 space is identified as $\mathrm{SU}(1,1) \times \mathrm{SU}(1,1) (\cong \mathrm{SL}(2;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})) \sim \mathrm{SO}(2,2)$ and the isometry of S^3 is identified as $\mathrm{SU}(2) \times \mathrm{SU}(2) \sim \mathrm{SO}(4)$. We denote the generators of $\mathrm{SU}(1,1)$ Lie algebra as $m^{\alpha}{}_{\beta}$ $(\alpha, \beta = 1, 2)$, and the generators of $\mathrm{SU}(2)$ Lie algebra as $m^{i}{}_{j}$ $(i, j = \hat{1}, \hat{2})$ by following the notation of [26]. (Here, we concentrate on the holomorphic sector and the anti-holomorphic sector can be defined in the similar way.) The commutation relations are given by

$$[m^{\alpha}_{\beta}, m^{\gamma}_{\delta}] = \delta^{\gamma}_{\beta} m^{\alpha}_{\delta} - \delta^{\alpha}_{\delta} m^{\gamma}_{\beta} \,, \qquad [m^{i}_{j}, m^{k}_{n}] = \delta^{k}_{j} m^{i}_{n} - \delta^{i}_{n} m^{k}_{j} \,, \tag{2.1}$$

and the hermitean conjugations are defined as

$$\begin{aligned} (m^1_1)^\dagger &= m^1_1 \,, \qquad & (m^2_1)^\dagger &= m^1_2 \,, \qquad & (m^1_2)^\dagger &= m^2_1 \,, \\ (m^{\hat{1}}_{\hat{1}})^\dagger &= m^{\hat{1}}_{\hat{1}} \,, \qquad & (m^{\hat{2}}_{\hat{1}})^\dagger &= -m^{\hat{1}}_{\hat{2}} \,, \qquad & (m^{\hat{1}}_{\hat{2}})^\dagger &= -m^{\hat{2}}_{\hat{1}} \,. \end{aligned} \tag{2.2}$$

The generators of odd sector q^{α}_{i} and q^{i}_{α} correspond to 8(+8) supercharges and the commutation relations are

$$\begin{aligned} [m^{\alpha}_{\beta}, q^{k}_{\gamma}] &= -\delta^{\alpha}_{\gamma} q^{k}_{\beta} + \frac{1}{2}\delta^{\alpha}_{\beta} q^{k}_{\gamma} \,, \qquad [m^{i}_{j}, q^{k}_{\alpha}] = \delta^{k}_{j} q^{i}_{\alpha} - \frac{1}{2}\delta^{i}_{j} q^{k}_{\alpha} \,, \\ [m^{i}_{j}, q^{\alpha}_{k}] &= -\delta^{i}_{k} q^{\alpha}_{j} + \frac{1}{2}\delta^{i}_{j} q^{\alpha}_{k} \,, \qquad [m^{\alpha}_{\beta}, q^{\gamma}_{k}] = \delta^{\gamma}_{\beta} q^{\alpha}_{k} - \frac{1}{2}\delta^{\alpha}_{\beta} q^{\gamma}_{k} \,, \\ \{q^{i}_{\alpha}, q^{\beta}_{j}\} &= i\left(\delta^{i}_{j} m^{\beta}_{\alpha} + \delta^{\beta}_{\alpha} m^{i}_{j}\right) . \end{aligned} \tag{2.3}$$

The hermitean conjugations are defined as

$$\begin{aligned} (q^1_{\hat{1}})^\dagger &= -i q^{\hat{1}}_1 \,, \quad & (q^2_{\hat{2}})^\dagger &= -i q^{\hat{2}}_2 \,, \quad & (q^2_{\hat{1}})^\dagger &= i q^{\hat{1}}_2 \,, \quad & (q^1_{\hat{2}})^\dagger &= i q^{\hat{2}}_1 \,, \\ (q^{\hat{1}}_1)^\dagger &= i q^1_{\hat{1}} \,, \quad & (q^{\hat{2}}_2)^\dagger &= i q^2_{\hat{2}} \,, \quad & (q^{\hat{2}}_1)^\dagger &= -i q^1_{\hat{2}} \,, \quad & (q^{\hat{1}}_2)^\dagger &= -i q^2_{\hat{1}} \,. \end{aligned} \tag{2.4}$$

Now, we take a light-cone basis and a Penrose limit. First, we consider the even sector. We redefine the generators as

$$
\begin{aligned}
J &= -m_1^1 + m_{\hat{1}}^{\hat{1}}\,, & F &= -\frac{1}{R}\left(m_1^1 + m_{\hat{1}}^{\hat{1}}\right)\,,\\
P_1 &= \frac{i}{\sqrt{R}} m_1^2\,, & P_1^* &= -\frac{i}{\sqrt{R}} m_2^1\,,\\
P_2 &= -\frac{i}{\sqrt{R}} m_{\hat{2}}^{\hat{1}}\,, & P_2^* &= -\frac{i}{\sqrt{R}} m_{\hat{1}}^{\hat{2}}\,,
\end{aligned}
\tag{2.5}
$$

and take the limit of $R \to \infty$. Then we obtain the commutation relations as

$$
[J, P_i] = P_i\,, \qquad [J, P_i^*] = -P_i^*\,, \qquad [P_i, P_j^*] = \delta_{ij} F\,,
\tag{2.6}
$$

which is the same as the ones of H_6 Lie algebra. The hermitean conjugations are given by

$$
(J)^\dagger = J\,, \qquad (F)^\dagger = F\,, \qquad (P_i)^\dagger = P_i^*\,, \qquad (P_i^*)^\dagger = P_i\,.
\tag{2.7}
$$

The analysis of odd part can be done just like the even part. We redefine the generates of odd sector as

$$
\begin{aligned}
Q^{--+} &= -\frac{i}{\sqrt{R}} q_{\hat{1}}^1\,, & Q^{+++} &= -\frac{i}{\sqrt{R}} q_{\hat{2}}^2\,,\\
Q^{++-} &= -\frac{1}{\sqrt{R}} q_1^{\hat{1}}\,, & Q^{---} &= -\frac{1}{\sqrt{R}} q_2^{\hat{2}}\,,\\
Q^{-++} &= q_{\hat{2}}^1\,, & Q^{+-+} &= q_{\hat{1}}^2\,,\\
Q^{-+-} &= i q_2^{\hat{1}}\,, & Q^{+--} &= i q_1^{\hat{2}}\,,
\end{aligned}
\tag{2.8}
$$

and take the limit of $R \to \infty$. The commutation relations with the generators of even sector becomes

$$
\begin{aligned}
[J, Q^{++a}] &= Q^{++a}\,, & [J, Q^{--a}] &= -Q^{--a}\,,\\
[P_1, Q^{-+a}] &= -Q^{++a}\,, & [P_1^*, Q^{+-a}] &= Q^{--a}\,,\\
[P_2, Q^{+-a}] &= -Q^{++a}\,, & [P_2^*, Q^{-+a}] &= -Q^{--a}\,,
\end{aligned}
\tag{2.9}
$$

and the other commutation relations vanish. The anti-commutation relations among odd sector are obtained as

$$
\begin{aligned}
\{Q^{--a}, Q^{++b}\} &= \epsilon^{ab} F\,, & \{Q^{-+a}, Q^{+-b}\} &= \epsilon^{ab} J\,,\\
\{Q^{++a}, Q^{+-b}\} &= \epsilon^{ab} P_1\,, & \{Q^{-+a}, Q^{--b}\} &= \epsilon^{ab} P_1^*\,,\\
\{Q^{-+a}, Q^{++b}\} &= \epsilon^{ab} P_2\,, & \{Q^{+-a}, Q^{--b}\} &= \epsilon^{ab} P_2^*\,.
\end{aligned}
\tag{2.10}
$$

The hermitean conjugations are given by $(Q^{\epsilon_1,\epsilon_2,\epsilon_3})^\dagger = Q^{-\epsilon_1,-\epsilon_2,-\epsilon_3}$. The authors of [8] obtained the superalgebras of the pp-wave backgrounds by contracting the superalgebras of $AdS_5 \times S^5$, $AdS_4 \times S^7$ and $AdS_7 \times S^4$. Our result is the counterpart of the case of $AdS_3 \times S^3$ and we obtain the supersymmetric extension of H_6 Lie algebra.

3. Superstring theory on pp-wave background with NSNS-flux

The superstring theory on the $AdS_3 \times S^3 (\times M^4)$ ($M^4 = T^4$ or $K3$) with NSNS-flux can be described by the $\mathrm{SL}(2;\mathbb{R}) \times \mathrm{SU}(2)$ super WZW model. The Penrose limit is realized by contracting these currents [27] and it becomes the WZW model whose target space is 6 dimensional Heisenberg group H_6. This kind of models were investigated in [28, 29, 30] and studied recently with newer motivations [7, 11, 13]. The sigma model approach of quantization was taken in [30, 7] and the current algebra method was developed in [28, 11] by using free field realization. Although these two approaches should be equivalent, we shall use the latter method, because the similarity with the analysis developed in [15] for the case of $AdS_3 \times S^3$ becomes more transparent.

3.1 H_6 super WZW model as a Penrose limit

The superstring theory on the $AdS_3 \times S^3$ can be described by the $\mathrm{SL}(2;\mathbb{R}) \times \mathrm{SU}(2)$ super WZW model. We set the level of each current algebra to an equal value $k \in \mathbb{Z}_{>0}$. (The bosonic parts of current algebras $\mathrm{SL}(2;\mathbb{R})$ and $\mathrm{SU}(2)$ have the levels $k+2$ and $k-2$, respectively). This system is described by the super current algebras

$$
\begin{aligned}
\mathcal{J}^A(z,\theta) &= \sqrt{\frac{k}{2}}\psi^A(z) + \theta J^A(z) \qquad (\text{for } \mathrm{SL}(2;\mathbb{R}))\,, \\
\mathcal{K}^a(z,\theta) &= \sqrt{\frac{k}{2}}\chi^a(z) + \theta K^a(z) \qquad (\text{for } \mathrm{SU}(2))\,.
\end{aligned} \tag{3.1}
$$

The "total currents" $J^A(z)$ and $K^a(z)$ ($A, a = \pm, 3$) satisfy the following operator product expansions (OPEs)

$$
\begin{aligned}
&J^3(z)J^3(w) \sim -\frac{k}{2(z-w)^2}\,, \qquad J^3(z)J^\pm(w) \sim \frac{\pm J^\pm(w)}{z-w}\,, \\
&J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} - \frac{2J^3(w)}{z-w}\,,
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
&K^3(z)K^3(w) \sim \frac{k}{2(z-w)^2}\,, \qquad K^3(z)K^\pm(w) \sim \frac{\pm K^\pm(w)}{z-w}\,, \\
&K^+(z)K^-(w) \sim \frac{k}{(z-w)^2} + \frac{2K^3(w)}{z-w}\,,
\end{aligned} \tag{3.3}
$$

and the free fermions ψ^A and χ^a are defined by the OPEs

$$
\begin{aligned}
&\psi^3(z)\psi^3(w) \sim -\frac{1}{z}\,, \qquad \psi^+(z)\psi^-(w) \sim \frac{2}{z-w}\,, \\
&\chi^3(z)\chi^3(w) \sim \frac{1}{z}\,, \qquad \chi^+(z)\chi^-(w) \sim \frac{2}{z-w}\,,
\end{aligned} \tag{3.4}
$$

and transformed by the action of the total currents as follows

$$
J^3(z)\psi^\pm(w) \sim \psi^3(z)J^\pm(w) \sim \frac{\pm\psi^\pm(w)}{z-w}\,,
$$

$$J^{\pm}(z)\psi^{\mp}(w) \sim \mp\frac{2\psi^3(w)}{z-w}, \tag{3.5}$$

$$K^3(z)\chi^{\pm}(w) \sim \chi^3(z)K^{\pm}(w) \sim \frac{\pm\chi^{\pm}(w)}{z-w},$$

$$K^{\pm}(z)\chi^{\mp}(w) \sim \pm\frac{2\chi^3(w)}{z-w}, \tag{3.6}$$

The Penrose limit of this model is given by a noncompact super WZW model associated with the 6 dimensional Heisenberg group H_6, which is a natural generalization of the Nappi-Witten model [28, 29, 30]. According to [27], we can obtain the supercurrents of this model by contracting those of the $\mathrm{SL}(2;\mathbb{R})\times\mathrm{SU}(2)$ model. We redefine the supercurrents (3.1) as

$$\begin{aligned}
\mathcal{J}(z,\theta) &= \mathcal{K}^3(z,\theta)+\mathcal{J}^3(z,\theta), & \mathcal{F}(z,\theta) &= \frac{1}{k}(\mathcal{K}^3(z,\theta)-\mathcal{J}^3(z,\theta)),\\
\mathcal{P}_1(z,\theta) &= \frac{1}{\sqrt{k}}\mathcal{J}^+(z,\theta), & \mathcal{P}_1^*(z,\theta) &= \frac{1}{\sqrt{k}}\mathcal{J}^-(z,\theta),\\
\mathcal{P}_2(z,\theta) &= \frac{1}{\sqrt{k}}\mathcal{K}^+(z,\theta), & \mathcal{P}_2^*(z,\theta) &= \frac{1}{\sqrt{k}}\mathcal{K}^-(z,\theta),
\end{aligned} \tag{3.7}$$

and take the limit $k\to\infty$ with keeping the eigenvalues of $K_0^3-J_0^3$ order $\mathcal{O}(k)$ but the eigenvalues of $K_0^3+J_0^3$ much smaller than k. Then we obtain the supercurrent algebra of the H_6 super WZW model as

$$\begin{aligned}
\mathcal{J}(\theta,z) &= \psi_J(z)+\theta J(z), & \mathcal{F}(\theta,z) &= \psi_F(z)+\theta F(z),\\
\mathcal{P}_i(\theta,z) &= \psi_{P_i}(z)+\theta P_i(z), & \mathcal{P}_i^*(\theta,z) &= \psi_{P_i^*}(z)+\theta P_i^*(z),
\end{aligned} \tag{3.8}$$

where $i=1,2$. The total currents $J(z)$, $F(z)$, $P_i(z)$ and $P_i^*(z)$ satisfy the OPEs

$$\begin{aligned}
J(z)P_i(w) &\sim \frac{P_i(w)}{z-w}, \qquad J(z)P_i^*(w) \sim -\frac{P_i^*(w)}{z-w},\\
P_i(z)P_j^*(w) &\sim \delta_{ij}\left(\frac{1}{(z-w)^2}+\frac{F(w)}{z-w}\right),\\
J(z)F(w) &\sim \frac{1}{(z-w)^2}.
\end{aligned} \tag{3.9}$$

Other OPEs have no singular terms. Their superpartners ψ_J, ψ_F, ψ_{P_i} and $\psi_{P_i^*}$ are free fermions defined by

$$\psi_{P_i}(z)\psi_{P_j^*}(w) \sim \frac{\delta_{ij}}{z-w}, \qquad \psi_J(z)\psi_F(w) \sim \frac{1}{z-w}. \tag{3.10}$$

The total currents J, F, P_i and P_i^* non-trivially act on these free fermions as

$$\begin{aligned}
J(z)\psi_{P_i}(w) &\sim \psi_J(z)P_i(w) \sim \frac{\psi_{P_i}}{z-w},\\
J(z)\psi_{P_i^*}(w) &\sim \psi_J(z)P_i^*(w) \sim -\frac{\psi_{P_i^*}}{z-w},\\
P_i(z)\psi_{P_j^*}(w) &\sim \psi_{P_i}(z)P_j^*(w) \sim \delta_{ij}\frac{\psi_F}{z-w}.
\end{aligned} \tag{3.11}$$

The physical meaning of this contraction is clear. In the context of AdS_3/CFT_2 correspondence, the eigenvalue of $-J_0^3$ corresponds to the conformal weight Δ (and also the space-time energy) in the boundary conformal theory and the eigenvalue of K_0^3 is interpreted as the space-time R-charge Q. Therefore we can expect that the above contraction amounts to focusing on the states of string theory on $AdS_3 \times S^3$ (or the boundary conformal theory) that are very close to the BPS bound and have large R-charges. We call them as "almost BPS states" according to [3] and they are characterized more precisely as

$$\Delta + Q \sim k \gg 1 \,, \qquad \Delta - Q \ll k \,. \tag{3.12}$$

The $\mathcal{N}=1$ superconformal symmetry is realized as follows. Because the total currents have non-trivial OPEs with fermions, it is convenient to introduce the "bosonic currents" which can be treated independently of fermions. They are defined as

$$\begin{aligned} \hat{J} &= J - \psi_{P_1}\psi_{P_1^*} - \psi_{P_2}\psi_{P_2^*} \,, & \hat{F} &= F \,, \\ \hat{P}_i &= P_i - \psi_F \psi_{P_i} \,, & \hat{P}_i^* &= P_i^* + \psi_F \psi_{P_i^*} \,, \end{aligned} \tag{3.13}$$

which again satisfy the same OPE (3.9) but have no singular OPEs with the free fermions. The $\mathcal{N}=1$ supercurrent is now defined in the standard fashion

$$\begin{aligned} G &= \hat{J}\psi_F + \hat{F}\psi_J + \sum_{i=1}^{2} (\hat{P}_i \psi_{P_i^*} + \hat{P}_i^* \psi_i + \psi_F \psi_{P_i} \psi_{P_i^*}) \\ &\equiv J\psi_F + F\psi_J + \sum_{i=1}^{2} (\hat{P}_i \psi_{P_i^*} + \hat{P}_i^* \psi_i) \,, \end{aligned} \tag{3.14}$$

and the total currents can be given by acting G on the fermions ψ_J, ψ_F, ψ_{P_i} and $\psi_{P_i^*}$ as it should be.

The analysis can be performed by using the abstract current algebra techniques in principle. However, it is easier to make use of the following free field realizations as given in [28]. We introduce the free bosons X^+, X^-, Z_i and Z_i^* $(i = 1, 2)$ defined by the OPEs

$$X^+(z)X^-(w) \sim -\ln(z-w) \,, \qquad Z_i(z)Z_j^*(w) \sim -\delta_{ij}\ln(z-w) \,, \tag{3.15}$$

and rewrite the fermions ψ_J, ψ_F, ψ_{P_i} and $\psi_{P_i^*}$ as

$$\psi_F = \psi^+ \,, \qquad \psi_J = \psi^- \,, \qquad \psi_{P_i} = \psi_i e^{iX^+} \,, \qquad \psi_{P_i^*} = \psi_i^* e^{-iX^+} \,, \tag{3.16}$$

where the new fermions are defined by

$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w} \,, \qquad \psi_i(z)\psi_j^*(w) \sim \frac{\delta_{ij}}{z-w} \,. \tag{3.17}$$

The total currents can be now expressed as

$$\begin{aligned} F &= i\partial X^+ \,, & J &= i\partial X^- \,, \\ P_i &= e^{iX^+}(i\partial Z_i + \psi^+\psi_i) \,, & P_i^* &= e^{-iX^+}(i\partial Z_i^* - \psi^+\psi_i^*) \,, \end{aligned} \tag{3.18}$$

and the bosonic currents are written as

$$\hat{F} = i\partial X^+ , \qquad \hat{J} = i\partial X^- - \psi_1\psi_1^* - \psi_2\psi_2^* - 2i\partial X^+ ,$$
$$\hat{P}_i = e^{iX^+} i\partial Z_i , \qquad \hat{P}_i^* = e^{-iX^+} i\partial Z_i^* . \tag{3.19}$$

In terms of these free fields the superconformal current is rewritten as the standard form of flat background

$$G = \psi^+ i\partial X^- + \psi^- i\partial X^+ + \psi_i^* i\partial Z_i + \psi_i i\partial Z_i^* . \tag{3.20}$$

We also introduce the free fields Y^i and λ^i ($i = 1,2,3,4$) to describe the remaining T^4 sector as

$$\lambda^i(z)\lambda^j(w) \sim \frac{\delta^{ij}}{z-w} , \qquad Y^i(z)Y^j(w) \sim -\delta^{ij} \ln(z-w) . \tag{3.21}$$

In order to analyze the space-time fermions, we have to introduce the spin fields, which are defined by using the bosonized fermions. We bosonize the fermions as

$$\begin{aligned}
\psi^{\pm 0} &= \psi^{\pm} = e^{\pm iH_0} , \\
\psi^{+j} &= \psi_j = e^{+iH_j} , \\
\psi^{-j} &= \psi_j^* = e^{-iH_j} , \\
\psi^{\pm 3} &= \frac{1}{\sqrt{2}}(\lambda^1 \pm i\lambda^2) = e^{\pm iH_3} , \\
\psi^{\pm 4} &= \frac{1}{\sqrt{2}}(\lambda^3 \pm i\lambda^4) = e^{\pm iH_4} ,
\end{aligned} \tag{3.22}$$

and define the spin fields as

$$S^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3\epsilon_4} = \exp\left(\frac{i}{2}\sum_{j=0}^{4} \epsilon_j H_j\right) . \tag{3.23}$$

The GSO condition imposes $\prod_{j=0}^4 \epsilon_j = +1$ in the convention of this paper. Precisely speaking, the OPEs including the spin fields are affected by the cocycle factors and they depend on the notation of gamma matrices. We summarize our convention of gamma matrices in appendix A.

3.2 Hilbert space of H_6 super WZW model

The irreducible representations of the current algebra of Nappi-Witten model (that is the H_4 WZW model) are classified in [28]. It is easy to extend to the case of H_6 super WZW model. We shall here focus on the "type II" representation (corresponding to the highest weight representation of the zero-mode subalgebra) in the terminology of [28], and later we discuss the other types of representations. The vacuum state (in the NS sector) is characterized by

$$\begin{aligned}
J_0|j,\eta\rangle &= j|j,\eta\rangle , & F_0|j,\eta\rangle &= \eta|j,\eta\rangle , \\
P_{i,n}|j,\eta\rangle &= 0 , \quad ({}^\forall n \geq 0) , & P_{i,n}^*|j,\eta\rangle &= 0 , \quad ({}^\forall n > 0) , \\
\Psi_r|j,\eta\rangle &= 0 , \quad \left({}^\forall r > 0 , \quad r \in \frac{1}{2} + \mathbb{Z}\right) , & &
\end{aligned} \tag{3.24}$$

where Ψ represents all the fermionic fields ψ_J, ψ_F, ψ_{P_i} and $\psi_{P_i^*}$. We assume that $j \in \mathbb{R}$ and $0 < \eta < 1$.

It is useful to rewrite this representation (3.24) in terms of the free fields $X^{\pm}$, Z_i, Z_i^*, $\psi^{\pm}$, ψ_i and ψ_i^*. This is nothing but a Fock representation with the Fock vacuum defined by the vertex operator

$$V = \exp\left(ijX^+ + i\eta X^-\right)\sigma_\eta\,, \tag{3.25}$$

where σ_η is the (chiral) twist field. This field imposes the boundary conditions

$$\begin{aligned} i\partial Z_i(e^{2\pi i}z) &= e^{-2\pi i\eta} i\partial Z_i(z)\,, \qquad & \psi_i(e^{2\pi i}z) &= e^{-2\pi i\eta}\psi_i(z)\,, \\ i\partial Z_i^*(e^{2\pi i}z) &= e^{2\pi i\eta} i\partial Z_i^*(z)\,, \qquad & \psi_i^*(e^{2\pi i}z) &= e^{2\pi i\eta}\psi_i^*(z)\,, \end{aligned} \tag{3.26}$$

which ensure the locality of H_6 supercurrents. More precisely, σ_η should have the following OPEs

$$\begin{aligned} i\partial Z_i(z)\sigma_\eta(w) &\sim (z-w)^{-\eta}{\tau^i}_\eta(w)\,, \qquad & i\partial Z_i^*(z)\sigma_\eta(w) &\sim (z-w)^{\eta-1}{\tau'^i}_\eta(w)\,, \\ \psi_i(z)\sigma_\eta(w) &\sim (z-w)^{-\eta}{t^i}_\eta(w)\,, \qquad & \psi_i^*(z)\sigma_\eta(w) &\sim (z-w)^{\eta}{t'^i}_\eta(w)\,, \end{aligned} \tag{3.27}$$

where ${\tau^i}_\eta$, ${\tau'^i}_\eta$, ${t^i}_\eta$ and ${t'^i}_\eta$ are the descendant twist fields. This twist operator σ_η has the conformal weight

$$h(\sigma_\eta) = 2\times\frac{1}{2}\eta(1-\eta) + 2\times\frac{1}{2}\eta^2 = \eta\,. \tag{3.28}$$

As already discussed in [28, 11], there is the spectral flow symmetry

$$\begin{aligned} J_n &\to J_n\,, \qquad & F_n &\to F_n + p\delta_{n,0}\,, \qquad & P_{i,n} &\to P_{i,n+p}\,, \qquad & P_{i,n}^* &\to P_{i,n-p}^*\,, \\ \psi_{J,r} &\to \psi_{J,r}\,, \qquad & \psi_{F,r} &\to \psi_{F,r}\,, \qquad & \psi_{P_i,r} &\to \psi_{P_i,r+p}\,, \qquad & \psi_{P_i^*,r} &\to \psi_{P_i^*,r-p}\,, \end{aligned} \tag{3.29}$$

and hence we also consider the "flowed representation" as the natural extensions of (3.24). The vacuum states are given by (where we use $p\in\mathbb{Z}$ as the spectral flow number)

$$\begin{aligned} J_0|j,\eta,p\rangle &= j|j,\eta,p\rangle\,, & & & F_0|j,\eta,p\rangle &= (\eta+p)|j,\eta,p\rangle\,, & & \\ P_{i,n}|j,\eta,p\rangle &= 0\,, & &(^\forall n\geq -p)\,, & P_{i,n}^*|j,\eta,p\rangle &= 0\,, & &(^\forall n > p)\,, \\ \psi_{J,r}|j,\eta,p\rangle &= 0\,, & &(^\forall r>0)\,, & \psi_{F,r}|j,\eta,p\rangle &= 0\,, & &(^\forall r>0)\,, \\ \psi_{P_i,r}|j,\eta,p\rangle &= 0\,, & &(^\forall r>-p)\,, & \psi_{P_i^*,r}|j,\eta,p\rangle &= 0\,, & &(^\forall r>p)\,, \end{aligned} \tag{3.30}$$

where $n\in\mathbb{Z}$ and $r\in 1/2+\mathbb{Z}$. The reason why we should introduce the flowed representations is clear. This spectral flow symmetry (3.29) is actually the counterpart of that on $\mathrm{SL}(2;\mathbb{R})\times\mathrm{SU}(2)$ WZW model and it is not difficult to confirm it directly by taking pp-wave limit (3.7). In the case of the AdS_3 strings, the necessity of flowed representations is well-known [25]. We will later focus on the sectors with non-zero spectral flow number p to realize the (almost) BPS states and discuss the correspondence with the symmetric orbifold theory. We can again realize the flowed representation (3.30) by means of the Fock representation, in which the Fock vacuum corresponds to the vertex operator

$$V = \exp\left(ijX^+ + i(\eta+p)X^-\right)\sigma_\eta\,. \tag{3.31}$$

A few comments are in order:

1. We also have other types of irreducible representations of H_6 current algebra. The "Type III" representations have the lowest weights of zero-mode subalgebra and the eigenvalues of F take $-1 < \eta < 0$. The "Type I" representations have neither highest weights nor lowest weights and the eigenvalues of F are $\eta = 0$. There are also their spectrally flowed representations. As in the case of SL$(2;\mathbb{R})$ WZW model, (see, for example, [25]) the spectral flow symmetry interchanges the type III representation with the type II representation. In fact, we can easily see that

$$\mathcal{H}^{(\mathrm{II})}_{j,\eta,p} \cong \mathcal{H}^{(\mathrm{III})}_{j,\eta-1,p+1}\,, \tag{3.32}$$

where $\mathcal{H}^{(\mathrm{II})}_{j,\eta,p}$ is the flowed type II representation defined by using the vacuum (3.30) and $\mathcal{H}^{(\mathrm{III})}_{j,\eta,p}$ is the flowed type III representation defined similarly. (We must also make the redefinitions of fermionic oscillators as $\psi'_{P_i,r} := \psi_{P_i,r+1}$ and $\psi'_{P_i^*,r} := \psi_{P_i^*,r-1}$ to equate the both sides of (3.32).) Therefore, we only have to consider the type II representation if assuming the spectral flow symmetry. We also note that the $p \geq 0$ representations correspond to the positive energy states and the $p < 0$ representations to the negative energy ones in the original AdS$_3$ string theory.

On the other hand, the type I representations seem to be the counterparts of the principal continuous series in the AdS$_3$ string theory. Because there are no twisted fields in this sector, the vacua of type I representations simply correspond to the next vertex operators

$$V = \exp\left(ipX^- + i(j+n)X^+ + ip_i^* Z_i + ip_i Z_i^*\right)\,, \qquad n \in \mathbb{Z}\,. \tag{3.33}$$

If we do not consider the spectral flow ($p = 0$), there are only trivial massless states with zero energy, that do not propagate along the transverse plane (Z_i, Z_i^* directions).[2] For the cases of flowed type I representations ($p \neq 0$), many physical states are possible. The spectra of light-cone energies in these sectors are continuous and the strings freely propagate along the transverse plane. It is quite natural to identify these sectors with the "long string sectors" in the AdS$_3$ string [25]. On the other hand, the type II (and type III) representation should correspond to the "short string sectors", because they are the counterparts of the discrete series in the AdS$_3$ string. The strings in these sectors cannot freely propagate along the transverse plane because the string coordinates Z_i, Z_i^* are twisted.

2. We here remark the equivalence

$$\bigoplus_{n\in\mathbf{Z}} \mathcal{H}^{(\mathrm{II})}_{j+\frac{n\eta}{p+\eta},\eta,p} \cong \bigoplus_{n\in\mathbf{Z}} \mathcal{F}_{j+\frac{n\eta}{p+\eta},\eta,p}\,, \tag{3.34}$$

[2]There is no such massless states in the case of principal continuous series in AdS$_3$ strings. This fact is due to the existence of the background charge term $Q \sim \sqrt{\frac{2}{k}}$. However, this term can be neglected under the large k limit. This is the reason why there are such extra massless states.

where we denote the right hand side as the Fock representation of free fields $X^{\pm}$, Z_i, Z_i^*, $\psi^{\pm}$, ψ_i and ψ_i^*. In this sense, we can reproduce the exact physical Hilbert space by using the free field representation with no subtlety, (for example, the treatment of screening charges in the case of AdS$_3$ strings) and this fact is a great advantage of taking the Penrose limit.

3.3 Physical vertices of superstring on pp-wave background

Now, we analyse the physical vertex operators. We shall concentrate on the short string sectors (type II representation) with positive energies ($p \geq 0$) for the time being, and we will discuss later the long string sectors. In order to construct the physical vertices, it is convenient to make use of the free field representation previously discussed. We fix the Fock vacuum as

$$|j,\eta,p\rangle = \sigma_\eta e^{ijX^+ + i(\eta+p)X^-}|0\rangle\,, \qquad 0<\eta<1\,, \qquad p\in\mathbb{Z}_{\geq 0}\,. \tag{3.35}$$

We construct the physical vertex operators explicitly by the so-called DDF operators in the covariant gauge. Here we introduce the superghosts (γ,β) or the bosonized ones (ϕ,ξ,η). The BRST charge has the standard form of the free superstring theory as

$$Q_{\mathrm{BRST}} = \oint \left[c\left(T - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi - \eta\partial\xi + \partial c b\right) + \eta e^{\phi}G - b\eta\partial\eta e^{2\phi}\right], \tag{3.36}$$

where T and G are the total stress tensor and the superconformal current constructed from the free fields $X^{\pm}$, Z_i, Z_i^*, Y^i, $\psi^{\pm}$, ψ_i, ψ_i^* and λ^i.

The most important vertex operators are the generators of space-time supersymmetry algebra (2.6), (2.9) and (2.10) considered above. The generators of bosonic part are quite easy. They are nothing but the zero-modes of world-sheet H_6 (total) currents;

$$\begin{aligned}
\mathcal{J} &= \oint \psi^- e^{-\phi} = \oint i\partial X^- = J_0\,,\\
\mathcal{F} &= \oint \psi^+ e^{-\phi} = \oint i\partial X^+ = F_0\,,\\
\mathcal{P}_i &= \oint \psi_i e^{iX^+} e^{-\phi} = \oint \left(i\partial Z_i + \psi^+\psi_i\right) e^{iX^+} = P_{i,0}\,,\\
\mathcal{P}_i^* &= \oint \psi_i^* e^{-iX^+} e^{-\phi} = \oint \left(i\partial Z_i^* - \psi^+\psi_i^*\right) e^{-iX^+} = P_{i,0}^*\,,
\end{aligned} \tag{3.37}$$

where we implicitly identify the operators by using the picture changing operator. The generators of fermionic part are obtained by using the spin fields (3.23) as (in the $(-1/2)$ picture)

$$\begin{aligned}
\mathcal{Q}^{++a} &= \oint S^{+++aa} e^{iX^+} e^{-\frac{\phi}{2}}\,, \qquad & \mathcal{Q}^{--a} &= \oint S^{+--aa} e^{-iX^+} e^{-\frac{\phi}{2}}\,,\\
\mathcal{Q}^{+-a} &= \oint S^{-+-aa} e^{-\frac{\phi}{2}}\,, & \mathcal{Q}^{-+a} &= \oint S^{--+aa} e^{-\frac{\phi}{2}}\,.
\end{aligned} \tag{3.38}$$

These operators manifestly BRST invariant and they can locally act on the Fock space associated with $|j,\eta,p\rangle$ (irrespective of the values of j, η and p). We can directly check that they generate the super pp-wave algebra (2.6), (2.9) and (2.10). In particular, we note that

$$\{\mathcal{Q}^{-+a}, \mathcal{Q}^{+-b}\} = \epsilon^{ab}\mathcal{J}\,, \qquad [\mathcal{J}, \mathcal{Q}^{\pm\mp a}] = 0\,, \tag{3.39}$$

which indicates that $\mathcal{Q}^{-+a}$ and $\mathcal{Q}^{+-a}$ play the role of supercharges with the "hamiltonian" $\mathcal{J}$.

In order to analyse the spectrum of physical states we further need to introduce the DDF operators. Recalling the analysis in AdS_3 string theory [15], it is quite natural to consider the "affine extension" of (3.37) and (3.38) as

$$\begin{aligned}
\mathcal{P}_{i,n} &= \frac{1}{\sqrt{p+\eta}} \oint \psi_i e^{i\frac{n+\eta}{p+\eta}X^+} e^{-\phi}\,, \\
\mathcal{P}^*_{i,n} &= \frac{1}{\sqrt{p+\eta}} \oint \psi_i e^{i\frac{n-\eta}{p+\eta}X^+} e^{-\phi}\,, \\
\mathcal{Q}_n^{++a} &= \frac{1}{\sqrt{p+\eta}} \oint S^{+++aa} e^{i\frac{n+\eta}{p+\eta}X^+} e^{-\frac{\phi}{2}}\,, \\
\mathcal{Q}_n^{--a} &= \frac{1}{\sqrt{p+\eta}} \oint S^{+--aa} e^{i\frac{n-\eta}{p+\eta}X^+} e^{-\frac{\phi}{2}}\,.
\end{aligned} \tag{3.40}$$

These operators are BRST invariant and locally act on the Fock space associated with $|j,\eta,p\rangle$ (of the fixed η and p). It is obvious that

$$\begin{aligned}
\sqrt{p+\eta}\mathcal{P}_{i,p} &= \mathcal{P}_i\,, & \sqrt{p+\eta}\mathcal{P}^*_{i,-p} &= \mathcal{P}^*_i\,, \\
\sqrt{p+\eta}\mathcal{Q}_p^{++a} &= \mathcal{Q}^{++a}\,, & \sqrt{p+\eta}\mathcal{Q}_{-p}^{--a} &= \mathcal{Q}^{--a}\,.
\end{aligned} \tag{3.41}$$

Note that the supercharges $\mathcal{Q}^{\pm\mp a}$ do not have such affine extensions because the BRST invariance cannot be preserved. The DDF operators (3.40) satisfy the following (anti-)commutation relations (up to the picture changing and BRST exact terms)

$$\begin{aligned}
[\mathcal{P}_{i,m}, \mathcal{P}^*_{j,n}] &= \frac{m+\eta}{p+\eta}\delta_{ij}\delta_{m+n,0}\,, & \{\mathcal{Q}_m^{--a}, \mathcal{Q}_n^{++b}\} &= \epsilon^{ab}\delta_{m+n,0}\,, \\
[\mathcal{J}, \mathcal{P}_{i,n}] &= \frac{n+\eta}{p+\eta}\mathcal{P}_{i,n}\,, & [\mathcal{J}, \mathcal{P}^*_{i,n}] &= \frac{n-\eta}{p+\eta}\mathcal{P}^*_{i,n}\,, \\
[\mathcal{J}, \mathcal{Q}_n^{++a}] &= \frac{n+\eta}{p+\eta}\mathcal{Q}_n^{++a}\,, & [\mathcal{J}, \mathcal{Q}_n^{--a}] &= \frac{n-\eta}{p+\eta}\mathcal{Q}_n^{--a}\,,
\end{aligned} \tag{3.42}$$

It is also useful to remark that $(\mathcal{P}_{i,n}, \mathcal{Q}_n^{++a})$ and $(\mathcal{P}^*_{i,n}, \mathcal{Q}_n^{--a})$ are the supermultiplets with respect to the supercharges $\mathcal{Q}^{+-a}$ and $\mathcal{Q}^{-+a}$. More precisely, we find the relations

$$\begin{aligned}
& \mathcal{P}_{1,n} \overset{\mathcal{Q}^{-+a}}{\longrightarrow} \mathcal{Q}_n^{++a} \overset{\mathcal{Q}^{-+(-a)}}{\longrightarrow} \mathcal{P}_{2,n} \overset{\mathcal{Q}^{-+*}}{\longrightarrow} 0 \\
0 \overset{\mathcal{Q}^{+-*}}{\longleftarrow} & \mathcal{P}_{1,n} \overset{\mathcal{Q}^{+-(-a)}}{\longleftarrow} \mathcal{Q}_n^{++a} \overset{\mathcal{Q}^{+-a}}{\longleftarrow} \mathcal{P}_{2,n}\,,
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
& \mathcal{P}^*_{1,n} \overset{\mathcal{Q}^{+-a}}{\longrightarrow} \mathcal{Q}_n^{--a} \overset{\mathcal{Q}^{+-(-a)}}{\longrightarrow} \mathcal{P}^*_{2,n} \overset{\mathcal{Q}^{+-*}}{\longrightarrow} 0 \\
0 \overset{\mathcal{Q}^{-+*}}{\longleftarrow} & \mathcal{P}^*_{1,n} \overset{\mathcal{Q}^{-+(-a)}}{\longleftarrow} \mathcal{Q}_n^{--a} \overset{\mathcal{Q}^{-+a}}{\longleftarrow} \mathcal{P}^*_{2,n}\,.
\end{aligned} \tag{3.44}$$

The explicit forms of the (anti-)commutation relations are summarized in appendix B.

In order to construct the remaining DDF operators for the T^4 directions, it is convenient to relabel the fermions λ^i as

$$
\begin{aligned}
\lambda^{+-} &= \frac{1}{\sqrt{2}}(\lambda^1 + i\lambda^2)\,, & \lambda^{-+} &= \frac{1}{\sqrt{2}}(-\lambda^1 + i\lambda^2)\,, \\
\lambda^{++} &= \frac{1}{\sqrt{2}}(-\lambda^3 - i\lambda^4)\,, & \lambda^{--} &= \frac{1}{\sqrt{2}}(-\lambda^3 + i\lambda^4)\,,
\end{aligned} \tag{3.45}
$$

and the free bosons $Y^{a\dot{a}}$ are defined similarly. These have the following OPEs

$$
\lambda^{a\dot{a}}(z)\lambda^{b\dot{b}}(w) \sim \frac{\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}}{z-w}\,, \qquad Y^{a\dot{a}}(z)Y^{b\dot{b}}(w) \sim -\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\ln(z-w)\,. \tag{3.46}
$$

The DDF operators can be then given by

$$
\begin{aligned}
\mathcal{A}_n^{a\dot{a}} &= \frac{1}{\sqrt{p+\eta}}\oint \lambda^{a\dot{a}} e^{i\frac{n}{p+\eta}X^+} e^{-\phi} = \frac{1}{\sqrt{p+\eta}}\oint i\partial Y^{a\dot{a}} e^{i\frac{n}{p+\eta}X^+}\,, \\
\mathcal{B}_n^{+\pm} &= -\frac{1}{\sqrt{p+\eta}}\oint S^{+-+\mp\pm} e^{i\frac{n}{p+\eta}X^+} e^{-\frac{\phi}{2}}\,, \\
\mathcal{B}_n^{-\pm} &= \frac{1}{\sqrt{p+\eta}}\oint S^{++-\mp\pm} e^{i\frac{n}{p+\eta}X^+} e^{-\frac{\phi}{2}}\,,
\end{aligned} \tag{3.47}
$$

and they satisfy

$$
\begin{aligned}
[\mathcal{A}_m^{a\dot{a}}, \mathcal{A}_n^{b\dot{b}}] &= \frac{m}{p+\eta}\delta_{m+n}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\,, & [\mathcal{B}_m^{\alpha\dot{a}}, \mathcal{B}_n^{\beta\dot{b}}] &= \delta_{m+n}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\,, \\
[\mathcal{J}, \mathcal{A}_n^{a\dot{a}}] &= \frac{n}{p+\eta}\mathcal{A}_n^{a\dot{a}}\,, & [\mathcal{J}, \mathcal{B}_n^{\alpha\dot{a}}] &= \frac{n}{p+\eta}\mathcal{B}_n^{\alpha\dot{a}}\,.
\end{aligned} \tag{3.48}
$$

The pair of $(\mathcal{A}_n^{a\dot{a}}, \mathcal{B}_n^{\alpha\dot{a}})$ again become supermultiplets with respect to $\mathcal{Q}^{\pm\mp a}$ (see appendix B for the more precise relations)

$$
\begin{aligned}
& \mathcal{A}_n^{a\dot{a}} \xrightarrow{\mathcal{Q}^{\mp\pm(-a)}} \mathcal{B}_n^{\pm\dot{a}} \xrightarrow{\mathcal{Q}^{\pm\mp(-a)}} \mathcal{A}_n^{(-a)\dot{a}} \xrightarrow{\mathcal{Q}^{**(-a)}} 0 \\
0 \xleftarrow{\mathcal{Q}^{**a}} & \mathcal{A}_n^{a\dot{a}} \xleftarrow{\mathcal{Q}^{\pm\mp a}} \mathcal{B}_n^{\pm\dot{a}} \xleftarrow{\mathcal{Q}^{\mp\pm a}} \mathcal{A}_n^{(-a)\dot{a}}\,.
\end{aligned} \tag{3.49}
$$

Finally we discuss whether or not the other vertex operators can be constructed from the remaining spin fields $S^{-\pm\pm\pm\mp}$ and $S^{-\pm\pm\mp\pm}$, which are allowed by the GSO condition. The locality condition requires the "dressing" of the factors like $e^{\pm i\frac{n+\eta}{p+\eta}X^+}$, however these factors are not compatible with the BRST invariance. Therefore, we conclude that $(\mathcal{P}_{i,n}, \mathcal{Q}_n^{++a})$, $(\mathcal{P}^*_{i,n}, \mathcal{Q}_n^{--a})$ and $(\mathcal{A}_n^{a\dot{a}}, \mathcal{B}_n^{\alpha\dot{a}})$ are the complete set of DDF operators.

3.4 Spectrum of physical states

Because we have presented the complete list of DDF operators, it is now easy to construct general physical states. More precisely speaking, we are interested in the physical states corresponding to the almost BPS states in the original $AdS_3 \times S^3$ superstring theory characterized by

$$
\Delta + Q \sim k \gg 1\,, \qquad \Delta - Q \ll k\,, \tag{3.50}
$$

where Δ represents the space-time energy, (measured by $-J_0^3$ in the original AdS_3 string theory) that is identified as the conformal weight of the boundary conformal field theory, and Q represents the space-time R-charge (measured by K_0^3). This condition is equivalent to[3]

$$\mathcal{F} \gtrsim 1\,, \qquad |\mathcal{J}| \ll k\,, \tag{3.51}$$

and all string excitations satisfy this condition in the pp-wave limit $k \to +\infty$. The BPS states correspond to the cases of $\mathcal{J} = 0$ and belong to the short multiplets of the superalgebra (3.37) and (3.38). The condition $\mathcal{F} \gtrsim 1$ only leads to the restriction $p \geq 1$ with respect to the spectral flow number p. (we only consider the positive energy states.)

We concentrate on the sectors with no momenta along T^4 direction for the time being. It is not difficult to write down the complete list of BPS states for each of the fixed p and η. In the NS sector, we obtain (where we focus on the left-mover only)

$$\begin{aligned}
|\omega^0;\eta,p\rangle &= \psi_{1,-\frac{1}{2}+\eta}|0,\eta,p\rangle \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}}\,,\\
|\omega^2;\eta,p\rangle &= \psi_{2,-\frac{1}{2}+\eta}|0,\eta,p\rangle \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}}\,,
\end{aligned} \tag{3.52}$$

and we have two more BPS states in the R-sector as

$$|\omega^{1\pm};\eta,p\rangle = (S^{-++\mp\pm})_{-\frac{5}{8}+\eta}|0,\eta,p\rangle \otimes ce^{-\frac{\phi}{2}}|0\rangle_{\mathrm{gh}}\,. \tag{3.53}$$

Here, we point out the next relation, which is useful for our discussion

$$\begin{aligned}
&|\omega^0;\eta,p\rangle \xrightarrow{\mathcal{B}_0^{+\dot{a}}} |\omega^{1\dot{a}};\eta,p\rangle \xrightarrow{\mathcal{B}_0^{+(-\dot{a})}} |\omega^2;\eta,p\rangle \xrightarrow{\mathcal{B}_0^{+*}} 0\\
0 \xleftarrow{\mathcal{B}_0^{-*}} &|\omega^0;\eta,p\rangle \xleftarrow{\mathcal{B}_0^{-(-\dot{a})}} |\omega^{1\dot{a}};\eta,p\rangle \xleftarrow{\mathcal{B}_0^{-\dot{a}}} |\omega^2;\eta,p\rangle\,.
\end{aligned} \tag{3.54}$$

Now, there is an obvious correspondence with the "chiral part" of the cohomology ring of T^4 by identifying $\mathcal{B}_0^{+\dot{a}}$ with the holomorphic one-form $dZ^{\dot{a}}$ on T^4. Therefore, emphasizing the correspondence to $H^*(T^4)$, the non-chiral BPS states can be explicitly written as

$$|\omega^{(q,\bar{q})};\eta,p\rangle = |\omega^q;\eta,p\rangle \otimes \overline{|\omega^{\bar{q}};\eta,p\rangle}\,, \qquad \left({}^\forall\omega^{(q,\bar{q})} \in H^{q,\bar{q}}(T^4)\right). \tag{3.55}$$

These states have the degenerate charge $\mathcal{F} = p + \eta$ for each sector of p and η.

We can construct the other types of physical states by making the DDF operators (3.40) and (3.47) act on these BPS states. We first note that the BPS states are actually the Fock vacua with respect to (3.40) and (3.47); that is

$$\begin{aligned}
\mathcal{P}_{i,n}|\omega;\eta,p\rangle &= 0\,, \quad ({}^\forall n \geq 0)\,, & \mathcal{P}^*_{i,n}|\omega;\eta,p\rangle &= 0\,, \quad ({}^\forall n > 0)\,,\\
\mathcal{Q}_n^{++a}|\omega;\eta,p\rangle &= 0\,, \quad ({}^\forall n \geq 0)\,, & \mathcal{Q}_n^{--a}|\omega;\eta,p\rangle &= 0\,, \quad ({}^\forall n > 0)\,,\\
\mathcal{A}_n^{a\dot{a}}|\omega;\eta,p\rangle &= 0\,, \quad ({}^\forall n \geq 0)\,, & \mathcal{B}_n^{\alpha\dot{a}}|\omega;\eta,p\rangle &= 0\,, \quad ({}^\forall n > 0)\,.
\end{aligned} \tag{3.56}$$

(For $\mathcal{B}_0^{\alpha\dot{a}}$, see (3.54).) We hence obtain

$$\begin{aligned}
&\mathcal{B}_{-n_1}^{\alpha_1\dot{a}_1}\cdots\mathcal{Q}_{-m_1}^{++b_1}\cdots\mathcal{Q}_{-k_1}^{--c_1}\cdots\otimes\bar{\mathcal{B}}_{-\bar{n}_1}^{\bar{\alpha}_1\bar{\dot{a}}_1}\cdots\bar{\mathcal{Q}}_{-\bar{m}_1}^{++\bar{b}_1}\cdots\bar{\mathcal{Q}}_{-\bar{k}_1}^{--\bar{c}_1}\cdots|\omega;\eta,p\rangle\,,\\
&n_i,\bar{n}_i,m_i,\bar{m}_i > 0\,, \qquad k_i,\bar{k}_i \geq 0\,, \qquad {}^\forall\omega \in H^*(T^4)\,,
\end{aligned} \tag{3.57}$$

[3] In our convention, the BPS inequality is equivalent to $\mathcal{J} \leq 0$.

as typical physical states. These states become almost BPS under the condition

$$|\mathcal{J}+\bar{\mathcal{J}}| \equiv \frac{1}{p+\eta}\left[\left(\sum_i n_i + \sum_i (m_i-\eta) + \sum_i (k_i+\eta)\right) + \right.$$
$$\left. + \left(\sum_i \bar{n}_i + \sum_i (\bar{m}_i-\eta) + \sum_i (\bar{k}_i+\eta)\right)\right] \ll k\,, \qquad (3.58)$$

which is always satisfied for sufficiently large k. Other states can be obtained by multiplying the supercharges $\mathcal{Q}^{\pm\mp a}$. In order to consider the level matching condition, we must keep it in mind that our free field representation corresponds to choosing different coordinate systems for the left and right movers as discussed in [11]. Hence we can include the non-vanishing "helicity in the transverse plane" $\mathcal{J}-\bar{\mathcal{J}} = h \in \mathbb{Z}$, and the level matching condition becomes

$$\left(\sum_i n_i + \sum_i (m_i-\eta) + \sum_i (k_i+\eta)\right) - \left(\sum_i \bar{n}_i + \sum_i (\bar{m}_i-\eta) + \sum_i (\bar{k}_i+\eta)\right) \in (p+\eta)\mathbb{Z}. \qquad (3.59)$$

We should note that under the limiting procedure $k\to\infty$, huge number of stringy excitations of the original AdS_3 superstring theory are included in our physical Hilbert space of string theory on pp-wave background. These states could correspond to very massive states, (which could possess very large energies $-J_0^3$) although they have the small $\mathcal{J}$-charges. This fact gives us a theoretical ground for making it possible to identify a lot of stringy excitations with the objects in the dual theory as in [3].

In order to complete our discussion we must also consider the sectors with non-trivial momenta along T^4. It is easy to show that there are no BPS states in these sectors. However, it is possible to construct the almost BPS states. We only consider a rectangular torus for simplicity and use R_a $(a=1,2,3,4)$ as the radii. The momenta of T^4 sector can be written as

$$p_a = \frac{n_a}{R_a} + \frac{w^a R_a}{2}\,, \qquad \bar{p}_a = \frac{n_a}{R_a} - \frac{w^a R_a}{2}\,, \qquad (n_a, w^a \in \mathbb{Z})\,, \qquad (3.60)$$

where n_a and w^a are the KK momenta and winding modes, respectively. Here we use the convention $\alpha' \equiv l_s^2 = 2$. The simplest physical states (in NSNS sector) have the next form

$$\psi_{i,-\frac{1}{2}+\eta}\bar{\psi}_{\bar{i},-\frac{1}{2}+\eta}|j,\bar{j},\eta,p;n_a,w^a\rangle \otimes c\bar{c}e^{-\phi-\bar{\phi}}|0\rangle_{\mathrm{gh}}\,, \qquad (3.61)$$

where $|j,\bar{j},\eta,p;n_a,w^a\rangle$ corresponds to the vertex operator

$$e^{i(jX^+ + (p+\eta)X^- + p_a Y^a)} \otimes e^{i(\bar{j}\bar{X}^+ + (p+\eta)\bar{X}^- + \bar{p}_a \bar{Y}^a)}\,. \qquad (3.62)$$

The on-shell condition leads to

$$j = -\frac{1}{2(p+\eta)}\sum_a p_a^2\,, \qquad \bar{j} = -\frac{1}{2(p+\eta)}\sum_a \bar{p}_a^2\,, \qquad (3.63)$$

and the level matching condition $j-\bar{j}\in\mathbb{Z}$ amounts to

$$\sum_a n_a w^a \in (p+\eta)\mathbb{Z}\,. \qquad (3.64)$$

The general physical states are obtained by making the DDF operators (3.40) and (3.47) and supercharges $\mathcal{Q}^{\pm\mp a}$ act on the above states (3.61). The condition for the almost BPS states is again given by

$$|\mathcal{J}+\bar{\mathcal{J}}| \ll k\,, \tag{3.65}$$

and the level matching condition is

$$\mathcal{J}-\bar{\mathcal{J}} \in \mathbb{Z}\,. \tag{3.66}$$

We have again huge number of stringy states for sufficiently large k.

Finally we make a few comments:

1. The spectrum of light-cone energies is given by $H_{\rm l.c.} = -(\mathcal{J}+\bar{\mathcal{J}})$. In particular, in the case of $p=0$ (although we are interested in the cases of $p \geq 1$), we can obtain

$$H_{\rm l.c.} = -(\mathcal{J}+\bar{\mathcal{J}}) = \frac{1}{\eta}(\mathrm{N}+\bar{\mathrm{N}}) + \mathrm{J} + \bar{\mathrm{J}} + \frac{1}{2\eta}\left(\sum_a p_a^2 + \sum_a \bar{p}_a^2\right), \tag{3.67}$$

where N and $\bar{\mathrm{N}}$ are the mode counting operators and J and $\bar{\mathrm{J}}$ are the "angular momentum operators". They act on the DDF operators as

$$\begin{aligned}
[\mathrm{N}, \mathcal{O}_n] &= -n\mathcal{O}_n\,, &&(\mathcal{O}_n = \mathcal{P}_{i,n}, \mathcal{P}^*_{i,n}, \mathcal{Q}_n^{++a}, \mathcal{Q}_n^{--a}, \mathcal{A}_n^{a\dot{a}}, \mathcal{B}_n^{\alpha\dot{a}})\,,\\
[\mathrm{J}, \mathcal{O}_n] &= -\mathcal{O}_n\,, &&(\mathcal{O}_n = \mathcal{P}_{i,n}, \mathcal{Q}_n^{++a})\,,\\
[\mathrm{J}, \mathcal{O}_n] &= \mathcal{O}_n\,, &&(\mathcal{O}_n = \mathcal{P}^*_{i,n}, \mathcal{Q}_n^{--a})\,,\\
[\mathrm{J}, \mathcal{O}_n] &= 0\,, &&(\mathcal{O}_n = \mathcal{A}_n^{a\dot{a}}, \mathcal{B}_n^{\alpha\dot{a}})\,.
\end{aligned} \tag{3.68}$$

The level matching condition is expressed as $\mathcal{J}-\bar{\mathcal{J}} \equiv h \in \mathbb{Z}$, which leads to the conditions $\mathrm{N}-\bar{\mathrm{N}} = 0$ and $\mathrm{J}-\bar{\mathrm{J}} \equiv h \in \mathbb{Z}$ for generic value of η, as discussed in [11]. This spectrum is consistent with the result given in [7] (and the appendix of [3]).

2. We can also construct many physical states in the sectors of spectrally flowed type I representations. The analysis is quite easy because it can be described by the usual free fields without twist operators. As we have already mentioned, it is plausible to suppose that they correspond to the long strings in the AdS_3 string theory [31, 32, 25]. More precisely, the strings in these sectors possess a continuous spectrum of the light-cone energies and can freely propagate along the transverse plane.

   The corresponding excitations do not seem to exist in the symmetric orbifold theory, which we will analyse in the next section, because it only includes the discrete spectrum. The existence of such continuous spectrum in the pp-wave strings reflects the non-compactness of the background, and is presumably related to the singularity of the type discussed in [32]. On the other hand, the symmetric orbifold theory corresponds to the "smooth point" in the moduli space of NS1-NS5 system with the non-vanishing world-sheet theta angle. From this reason the absence of the continuous spectrum does not indicate a contradiction. The detailed analysis of such singularity and the aspects of long string sectors will be significant subjects for our future study, however we shall not discuss them any more in this paper.

4. Comparison with $\mathrm{Sym}^M(T^4)$ σ-model

The well-known candidate of the dual theory of the superstrings on the $AdS_3 \times S^3 \times T^4$ is the $\mathcal{N} = (4,4)$ non-linear σ-model on the symmetric orbifold space $\mathrm{Sym}^M(T^4) \equiv (T^4)^M/S_M$, where $M = Q_1 Q_5$, for sufficiently large Q_1 and Q_5 [16, 17]. (Q_1 and Q_5 are the NS1 and NS5 charges, respectively.) The charge Q_5 can be identified as the level k of $\mathrm{SL}(2;\mathbb{R})$ and SU(2) WZW models. The charge Q_1 should be the implicit upper bound of spectral flow number p. (See, for example, [19]. Of course, p should not be bounded from the viewpoints of perturbative string theory. However, we take anyway the large Q_1 limit as well as large Q_5.) The main subject of this section is to analyse the spectrum of BPS and almost BPS states with large R-charges $Q(\gtrsim Q_5 \gg 1)$, and compare the spectrum of short string sectors with positive energies, which was studied in the previous section. To this aim, we start with a short review of $\mathrm{Sym}^M(T^4)$ σ-Model.

4.1 Short review of $\mathrm{Sym}^M(T^4)$ σ-model

The $\mathcal{N} = (4,4)$ superconformal field theory defined by the supersymmetric σ-model on symmetric orbifold $\mathrm{Sym}^M(T^4) \equiv (T^4)^M/S_M$ is described as follows. We use $4M$ free bosons $X^{a\dot{a}}_{(A)}$ $(A = 0,1,2,\ldots,M-1, a, \dot{a} = \pm)$ and free fermions $\Psi^{\alpha\dot{a}}_{(A)}$ and $\bar{\Psi}^{\dot{\alpha}\dot{a}}_{(A)}$ as the fundamental fields. The superconformal symmetry is realized by the following currents (where we only write the left-mover)

$$\begin{aligned}
T(z) &= -\frac{1}{2}\sum_A \epsilon_{ab}\,\epsilon_{\dot{a}\dot{b}}\,\partial X^{a\dot{a}}_{(A)}\,\partial X^{b\dot{b}}_{(A)} - \frac{1}{2}\sum_A \epsilon_{\alpha\beta}\,\epsilon_{\dot{a}\dot{b}}\,\Psi^{\alpha\dot{a}}_{(A)}\,\partial\Psi^{\beta\dot{b}}_{(A)}\,,\\
G^{\alpha a}(z) &= i\sum_A \epsilon_{\dot{a}\dot{b}}\,\Psi^{\alpha\dot{a}}_{(A)}\,\partial X^{a\dot{b}}_{(A)}\,,\\
K^{\alpha\beta}(z) &= -\frac{1}{2}\sum_A \epsilon_{\dot{a}\dot{b}}\,\Psi^{\alpha\dot{a}}_{(A)}\,\Psi^{\beta\dot{b}}_{(A)}\,.
\end{aligned}\tag{4.1}$$

These generate the $N = 4$ (small) superconformal algebra (SCA) with central charge $c = 6M$. In our convention, we set $\epsilon^{+-} = \epsilon_{-+} = 1$ and $\epsilon^{-+} = \epsilon_{+-} = -1$, and the OPEs of free fields are written as

$$X^{a\dot{a}}_{(A)}(z)X^{b\dot{b}}_{(B)}(0) \sim -\delta_{AB}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\ln z\,, \qquad \Psi^{\alpha\dot{a}}(z)\Psi^{\beta\dot{b}}(0) \sim \delta_{AB}\frac{\epsilon^{\alpha\beta}\epsilon^{\dot{a}\dot{b}}}{z}\,. \tag{4.2}$$

The usual convention of SU(2) current is given by $K^3 = K^{+-} = K^{-+}$ and $K^{\pm} = \pm K^{\pm\pm}$. The subalgebra of "zero-modes" $\{L_{\pm 1},\, L_0,\, G^{\alpha a}_{\pm 1/2},\, K^{\alpha\beta}_0\}$ and the counterpart of the right mover compose the super Lie algebra $\mathrm{PSU}(1,1|2)_L \times \mathrm{PSU}(1,1|2)_R$. This subalgebra corresponds to the supersymmetric algebra on the $AdS_3 \times S^3$ geometry, as we already mentioned.

According to the general approach to the orbifold conformal field theory [33], we have various twisted sectors corresponding to the each element g of S_M. The Hilbert space of each twisted sector is defined with the following boundary condition ($z \equiv e^{\tau + i\sigma}$) as

$$\Phi_{(A)}(\tau, \sigma + 2\pi) = \Phi_{g(A)}(\tau, \sigma)\,, \tag{4.3}$$

where $\Phi_{(A)}(\tau,\sigma)$ represents $X^{a\dot{a}}_{(A)}$, $\Psi^{\alpha\dot{a}}_{(A)}$ and $\bar{\Psi}^{\dot{\alpha}\dot{a}}_{(A)}$. We should take the projection onto the S_M-invariant subspace.

Because an arbitrary permutation can be decomposed by cyclic permutations, it is standard to label each twisted sector by a Young tableau $(N_1, \ldots, N_l)$, in which each row corresponds to the $\mathbb{Z}_{N_i}$-twisted sector. In the context of AdS_3/CFT_2 correspondence, (see, for example, [17]) each $\mathbb{Z}_{N_i}$-twisted sector describes a single-particle state and $(N_1, \ldots, N_l)$-twisted sector describes the Hilbert space of l-particle states. In order to compare with the physical spectrum of first quantized superstring theory, it is enough to focus on the single-particle Hilbert space. Hence, we shall concentrate on the $\mathbb{Z}_N$-twisted sector ($N \leq M$) from now on.

We label the objects in this sector by the index $\mathcal{A} \equiv [(A_0), \ldots, (A_{N-1})]$, where $A_i = 0, \ldots, N-1$ such that $A_i \neq A_j$ for $^\forall i \neq j$. The string coordinates of this sector are defined on the world-sheet $0 \leq \sigma \leq 2\pi$, which is rescaled from the N-times one $0 \leq \sigma \leq 2\pi N$. These coordinates are given by

$$\begin{aligned} &\Phi_{\mathcal{A}}(\tau, \sigma) = \Phi_{(A_r)}(\tau, N\sigma - 2\pi r) , \\ &\frac{2\pi r}{N} \leq \sigma \leq \frac{2\pi(r+1)}{N} , \qquad r = 0, 1, \ldots, N-1 , \end{aligned} \tag{4.4}$$

where Φ represents the fields $X^{a\dot{a}}$, $\Psi^{\alpha\dot{a}}$ and $\bar{\Psi}^{\dot{\alpha}\dot{a}}$. These variables $X^{a\dot{a}}_{(\mathcal{A})}$, $\Psi^{\alpha\dot{a}}_{(\mathcal{A})}$ and $\bar{\Psi}^{\dot{\alpha}\dot{a}}_{(\mathcal{A})}$ can be used to construct the $\mathcal{N} = 4$ superconformal currents $\{L_{(\mathcal{A}),n}, G^{\alpha a}_{(\mathcal{A}),r}, K^{\alpha\beta}_{(\mathcal{A}),n}\}$ with the central charge $c = 6$ in the manner similar to (4.1). However, we have to impose the $\mathbb{Z}_N$-invariance condition on the physical Hilbert space as

$$L_{(\mathcal{A}),\,0} - \bar{L}_{(\mathcal{A}),\,0} \in N\mathbb{Z} . \tag{4.5}$$

The superconformal currents compatible with this condition (4.5), which can act on the physical Hilbert space *independently of the right movers*, consists only of the modes of $n \in N\mathbb{Z}$.[4] More precisely, the superconformal currents describing properly the $\mathbb{Z}_N$-twisted sector $\{\hat{L}_n, \hat{G}^{\alpha a}_r, \hat{K}^{\alpha\beta}_n\}$ should be defined as follows [34, 35] (from now on, we shall omit the label $\mathcal{A}$ for simplicity and only present the NS sector)

$$\begin{aligned} \hat{L}_n &= \frac{1}{N} L_{nN} + \frac{N^2-1}{4N}\delta_{n0} , \\ \hat{G}^{\alpha a}_r &= \begin{cases} \frac{1}{\sqrt{N}} G^{\alpha a\,(NS)}_{Nr} , & (N = 2q+1) , \\ \frac{1}{\sqrt{N}} G^{\alpha a\,(R)}_{Nr} , & (N = 2q) , \end{cases} \\ \hat{K}^{\alpha\beta}_n &= K^{\alpha\beta}_{nN} . \end{aligned} \tag{4.6}$$

One can check directly that these operators generate the $\mathcal{N} = 4$ superconformal algebra with $c = 6N$. The anomaly term in the expression of $\hat{L}_n$ corresponds essentially to the Schwarzian derivative of the conformal mapping $z \longmapsto z^N$. We should note that the modes of hatted currents are counted by $\hat{L}_0$.

[4] The fractional modes, e.g., $\hat{L}_{\frac{n}{N}} \equiv \frac{1}{N} L_n$ (where $n \notin N\mathbb{Z}$), can also act on the physical Hilbert space. However, in that case, the left and right movers are not independent and restricted by the condition (4.5).

From these definitions (4.6) we can find that the vacuum $|0;N\rangle$ of $\mathbb{Z}_N$-twisted sector possesses the following properties:

- $N = 2q+1$

$$\begin{aligned} \hat{L}_0|0;N\rangle &= \frac{N^2-1}{4N}|0;N\rangle\,, \\ \hat{K}_0^3|0;N\rangle &= 0\,. \end{aligned} \tag{4.7}$$

- $N = 2q$

$$\begin{aligned} \hat{L}_0|0;N\rangle &= \left(\frac{N^2-1}{4N}+\frac{1}{4N}\right)|0;N\rangle \equiv \frac{N}{4}|0;N\rangle\,, \\ \hat{K}_0^3|0;N\rangle &= -\frac{1}{2}|0;N\rangle\,. \end{aligned} \tag{4.8}$$

When N is even, the supercurrent $\hat{G}_r^{\alpha a}$ in the NS sector is made of the one in the R sector before imposing the $\mathbb{Z}_N$-invariance. The extra vacuum energy "$\frac{1}{4N}$" and the extra R-charge "$-1/2$" in the case of $N = 2q$ originate from this fact.

Now, we focus on the BPS states (chiral primary states) in this $\mathbb{Z}_N$-twisted sector, which are defined by the next conditions

$$\begin{aligned} \hat{L}_n|\alpha\rangle &= \bar{\hat{L}}_n|\alpha\rangle = 0\,, && ({}^\forall n \geq 1)\,, \\ \hat{G}_r^{+a}|\alpha\rangle &= \bar{\hat{G}}_r^{+a}|\alpha\rangle = 0\,, && ({}^\forall r \geq -\frac{1}{2})\,, \\ \hat{G}_r^{-a}|\alpha\rangle &= \bar{\hat{G}}_r^{-a}|\alpha\rangle = 0\,, && ({}^\forall r \geq \frac{1}{2})\,. \end{aligned} \tag{4.9}$$

These conditions lead inevitably to

$$(\hat{L}_0 - \hat{K}_0^3)|\alpha\rangle = 0\,. \tag{4.10}$$

At this point it is not difficult to present the explicit forms of all the possible BPS states in the $\mathbb{Z}_N$-twisted sector. (See, for example, [36].) They are written as

$$|\omega^{(q,\bar{q})};N\rangle = |\omega^q;N\rangle \otimes \overline{|\omega^{\bar{q}};N\rangle}\,, \qquad {}^\forall\omega^{(q,\bar{q})} \in H^{q,\bar{q}}(T^4)\,, \qquad (q,\bar{q} = 0,1,2)\,, \tag{4.11}$$

where the left(right)-moving parts are defined by

- $N = 2q+1$

$$\begin{aligned} |\omega^0;N\rangle &= \prod_{i=0}^{q-1} \Psi^{++}_{-(\frac{1}{2N}+\frac{i}{N})} \Psi^{+-}_{-(\frac{1}{2N}+\frac{i}{N})}|0;N\rangle\,, \\ |\omega^{1\dot{a}};N\rangle &= \Psi^{+\dot{a}}_{-\frac{1}{2}}|\omega^0;N\rangle\,, \\ |\omega^2;N\rangle &= \Psi^{++}_{-\frac{1}{2}}\Psi^{+-}_{-\frac{1}{2}}|\omega^0;N\rangle\,. \end{aligned} \tag{4.12}$$

- $N = 2q$

$$
\begin{aligned}
|\omega^0; N\rangle &= \prod_{i=0}^{q-1} \Psi^{++}_{-\frac{i}{N}} \Psi^{+-}_{-\frac{i}{N}} |0; N\rangle \,, \\
|\omega^{1\dot{a}}; N\rangle &= \Psi^{+\dot{a}}_{-\frac{1}{2}} |\omega^0; N\rangle \,, \\
|\omega^2; N\rangle &= \Psi^{++}_{-\frac{1}{2}} \Psi^{+-}_{-\frac{1}{2}} |\omega^0; N\rangle \,.
\end{aligned} \tag{4.13}
$$

It is easy to check that

$$
\begin{aligned}
\hat{L}_0 |\omega^{(q,\bar{q})}; N\rangle &= \hat{K}^3_0 |\omega^{(q,\bar{q})}; N\rangle = \frac{q+N-1}{2} |\omega^{(q,\bar{q})}; N\rangle \,, \\
\hat{\bar{L}}_0 |\omega^{(q,\bar{q})}; N\rangle &= \hat{\bar{K}}^3_0 |\omega^{(q,\bar{q})}; N\rangle = \frac{\bar{q}+N-1}{2} |\omega^{(q,\bar{q})}; N\rangle \,.
\end{aligned} \tag{4.14}
$$

4.2 "pp wave limit" of $\mathrm{Sym}^M(T^4)$ σ-model and comparison with the superstring spectrum

Now, we focus on the (almost) BPS states with large R-charges. As we observed in (4.14), $\hat{L}_0 + \hat{K}^3_0$ takes the value of order $\mathcal{O}(N)$ for the BPS states in the $\mathbb{Z}_N$-twisted sector. In the context of AdS_3/CFT_2 correspondence, $\hat{L}_0$ and $\hat{K}^3_0$ are identified with $-J^3_0$ and K^3_0, respectively, which are the zero-modes of the total currents in $\mathrm{SL}(2;\mathbb{R})$ and $\mathrm{SU}(2)$ super WZW models. Recalling the relations (3.7), we must take the identification

$$
\mathcal{J} \longleftrightarrow \hat{K}^3_0 - \hat{L}_0 \,, \qquad \mathcal{F} \longleftrightarrow \frac{1}{k}(\hat{K}^3_0 + \hat{L}_0) \,. \tag{4.15}
$$

Therefore we should consider the case in which $N \sim k \gg 1$. We should also assume that $|\hat{K}^3_0 - \hat{L}_0| \ll k$. In this situation, it is very useful to introduce the next "pp-wave limit". We redefine

$$
\begin{aligned}
\mathcal{B}^{\pm\dot{a}\,\prime}_n &= \Psi^{\pm\dot{a}}_{\mp\frac{1}{2}+\frac{nk}{N}} \,, & \mathcal{A}^{a\dot{a}\,\prime}_n &= \frac{1}{\sqrt{N}} i\partial X^{a\dot{a}}_{\frac{nk}{N}} \,, \\
\mathcal{P}^{(l)\,\prime}_{1,n} &= -\frac{1}{\sqrt{N}} \left\{ \frac{N}{nk+l} \hat{L}_{\frac{nk+l}{N}} - \left(\frac{N}{nk+l} - 1 \right) \hat{K}^3_{\frac{nk+l}{N}} \right\} \,, \\
\mathcal{P}^{*(l)\,\prime}_{1,n} &= \frac{1}{\sqrt{N}} \left\{ \frac{N}{nk-l} \hat{L}_{\frac{nk-l}{N}} - \left(\frac{N}{nk-l} + 1 \right) \hat{K}^3_{\frac{nk-l}{N}} \right\} \,, \\
\mathcal{P}^{(l)\,\prime}_{2,n} &= \frac{1}{\sqrt{N}} \hat{K}^+_{-1+\frac{nk+l}{N}} \,, & \mathcal{P}^{*(l)\,\prime}_{2,n} &= \frac{1}{\sqrt{N}} \hat{K}^-_{1+\frac{nk-l}{N}} \,, \\
\mathcal{Q}^{++\pm(l)\,\prime}_n &= \pm \frac{\sqrt{N}}{nk+l} \hat{G}^{+\pm}_{-\frac{1}{2}+\frac{nk+l}{N}} \,, & \mathcal{Q}^{--\pm(l)\,\prime}_n &= \mp \frac{\sqrt{N}}{nk-l} \hat{G}^{-\pm}_{\frac{1}{2}+\frac{nk-l}{N}} \,, \\
\mathcal{Q}^{+-\pm\,\prime} &= \pm \hat{G}^{-\pm}_{1/2} \,, & \mathcal{Q}^{-+\pm\,\prime} &= \pm \hat{G}^{+\pm}_{-1/2} \,,
\end{aligned} \tag{4.16}
$$

where l runs over the range $l = 1, 2, \ldots, k-1$, and take the large k limit with keeping $\mathcal{F}$ finite. Remarkably, these operators satisfy the same (anti-)commutation relations as (3.42) and (3.48) with the identification $N = pk + l$ and $\eta = l/k$ under the large k limit. Strictly speaking, the mode expansions of twisted σ-model coordinates $i\partial X^{a\dot{a}}$ and $\Psi^{\alpha\dot{a}}$ depend on whether N is even or odd. (Recall the discussions in the previous subsection.) However, we can neglect safely the difference $1/2N$ under the assumption of large-N. It is quite important to note that there is the equal number of degrees of freedom after taking such large-N limit. The Hilbert space of $\mathbb{Z}_N$-twisted sector is spanned by the free oscillators $\Psi^{\pm\dot{a}}_{\mp 1/2+n/N}$ and $i\partial X^{a\dot{a}}_{n/N}$. For each energy level, there are the equal number of bosonic and fermionic oscillators defined in (4.16).

At this stage, we can compare the spectra of BPS and almost BPS states with the superstring spectrum. As already mentioned, the spectrum should be compared with the short string sectors with positive $p \geq 1$. We again forget the sectors with momenta along T^4 sector for the time being.

First, we consider the BPS states. We start with the simple consideration of degrees of freedom. At least with respect to the BPS states, we can expect that the equal number of physical states exist in both of the superstring theory on $AdS_3 \times S^3 \times T^4$ and the pp-wave background. This statement is valid as long as the pp-wave string theory is defined by the contraction (3.7). Fix the spectral flow number $p (\geq 1)$. It is known [22, 23] that, roughly speaking, there are about $k \times \dim H^*(T^4)$ BPS states with the R-charges $\frac{1}{2}kp \lesssim Q \lesssim \frac{1}{2}k(p+1)$ in the string theory on the $AdS_3 \times S^3 \times T^4$. This values of R-charges amount to $\mathcal{F} = p + l/k$ $(0 \leq l \leq k)$, for each of the spectrally flowed sector. Therefore, we should assume $\eta = l/k$ $(l = 1, 2, \ldots, k-1)$, so that the following physical Hilbert space

$$\mathcal{H}^{(p)}_{\text{pp-wave}} = \bigoplus_l \mathcal{H}_{\text{pp-wave}}(p, \eta = l/k) \tag{4.17}$$

includes the equal number of BPS states as those of $AdS_3 \times S^3 \times T^4$ superstring theory.[5]

Turning to the symmetric orbifold, we consider the following direct sum of the single particle Hilbert spaces of the $\mathbb{Z}_{N(l)}$ twisted sectors, (where we set $N(l) \equiv kp + l$ $(0 < l < k)$)

$$\mathcal{H}^{(p)}_{\text{symm}} = \bigoplus_l \mathcal{H}_{\text{symm}}(N(l) = kp + l)\,. \tag{4.18}$$

It is obvious that $\mathcal{H}^{(p)}_{\text{symm}}$ and $\mathcal{H}^{(p)}_{\text{pp-wave}}$ have the equivalent spectrum of BPS states. We also point out that the identification $\mathcal{B}^{+\dot{a}'}_0 (\equiv \Psi^{+\dot{a}}_{-1/2}) = \mathcal{B}^{+\dot{a}}_0$ is consistent with the relation between (3.54) and (4.12), (4.13).

Next, we consider the almost BPS states. Once we admit the correspondence of $\eta = l/k$, it is not difficult to find the almost BPS states in $\mathcal{H}^{(p)}_{\text{symm}}$ that correspond to the stringy excitations of the string theory side. First, we rewrite the DDF operators (3.40) and (3.47), which act on the Hilbert space $\mathcal{H}_{\text{pp-wave}}(p, \eta = l/k)$, as $\mathcal{P}^{(l)}_{i,n}$, $\mathcal{P}^{*(l)}_{i,n}$ and so on. We also rewrite the operators (4.16) as $\mathcal{P}^{(l,j)'}_{i,n}$, $\mathcal{P}^{*(l,j)'}_{i,n}$ and so on, for each $\mathbb{Z}_{N(j)}$-twisted sector $\mathcal{H}_{\text{symm}}(N(j))$ (where $1 \leq l \leq k-1$ and $1 \leq j \leq k-1$). The "diagonal terms" (e.g., $\mathcal{P}^{(l,l)'}_{i,n}$) generate the superalgebra equivalent to that of DDF operators $\mathcal{P}^{(l)}_{i,n}$, $\mathcal{P}^{*(l)}_{i,n}$

[5] The threshold values $l = 0$ and k cannot be included because there is a restriction $0 < \eta < 1$ in the type II representations. Moreover, the states with $l = k-1$ corresponds to the missing states in the original $AdS_3 \times S^3$ string theory for each p. (See, for example, [37, 22].) One might worry that η should be continuous in principle. These subtleties are, however, harmless for sufficiently large k.

and so on, as we already mentioned. Now the correspondence is obvious. The non-trivial consistency check is only the level matching condition. In the symmetric orbifold theory side, the level matching condition is given by

$$\hat{L}_0 - \bar{\hat{L}}_0 \in \mathbb{Z}\,. \tag{4.19}$$

This condition is consistent with the result of string theory side (3.59) under the above correspondence of DDF operators.

However, we should note here that the string Hilbert space $\mathcal{H}^{(p)}_{\text{pp-wave}}$ is strictly smaller than that of symmetric orbifold; that is to say

$$\mathcal{H}^{(p)}_{\text{pp-wave}} \overset{\subset}{\neq} \mathcal{H}^{(p)}_{\text{symm}}\,. \tag{4.20}$$

In fact, the “non-diagonal terms” (e.g., $\mathcal{P}^{(l,j)'}_{i,n}$ with $l \neq j$) have no counterparts in the string side. Therefore, we cannot define the corresponding DDF operators as local operators on the string Hilbert space. These missing states in the string spectrum may be compensated by non-perturbative excitations. Because we are now assuming the small string coupling, the non-perturbative excitations usually become very massive. Under the assumption of large k, the space of almost BPS states can include such very massive excitations in principle. However, our world-sheet analysis as a perturbative string theory cannot include such excitations. This is the reason why there are many missing states in the string side. The non-perturbative analysis will be an important task for future study.

Finally we consider the sectors with non-vanishing momenta. As in the string spectrum, there are no BPS states in these sectors, however there are many almost BPS states. Recall that $\mathcal{J} = \hat{K}^3_0 - \hat{L}_0$ and $\hat{L}_0 = \frac{1}{N}L_0 + \frac{N^2-1}{4N}$. The operator L_0 includes the contribution of momenta with the standard normalization. Hence, we find that the contributions of momenta to $\mathcal{J}$ and $\bar{\mathcal{J}}$ are given as follows (for the sector $\mathcal{H}_{\text{symm}}(N(l))$ with $N(l) \equiv kp + l$)

$$\Delta\mathcal{J} = -\frac{1}{2N(l)}\sum_a \left(\frac{n'_a}{R_a} + \frac{w'^a R_a}{2}\right)^2\,, \qquad \Delta\bar{\mathcal{J}} = -\frac{1}{2N(l)}\sum_a \left(\frac{n'_a}{R_a} - \frac{w'^a R_a}{2}\right)^2\,, \tag{4.21}$$

where n'_a and w'^a are KK momenta and winding modes as before. The level matching condition for the vacuum state can be read as

$$\sum_a n'_a w'^a \in N(l)\mathbb{Z}\,. \tag{4.22}$$

By comparing the analysis given in the last section (under the identification $\eta = l/k$), we obtain the following correspondence between the spectrum of string theory and that of the symmetric orbifold theory as

$$n'_a = \sqrt{k}n_a\,, \qquad w'^a = \sqrt{k}w^a\,, \tag{4.23}$$

where n_a and w^a are the zero-momenta in the string theory side.[6]

[6]Although $\sqrt{k}$ is not an integer in general, we can approximate it by $[\sqrt{k}]$, because we are now assuming large k, such as $|\sqrt{k} - [\sqrt{k}]|/\sqrt{k} \ll 1$.

We have observed that there are again many missing states in the string theory side. The essentially same aspect was already pointed out in the context of string theory on $AdS_3 \times S^3 \times T^4$ [20]. It may be worthwhile to comment on how such discrepancy is removed if assuming the fractional string excitations. These excitations do not exist in the perturbative string spectrum and may be at least explained in the S-dual picture. The existence of $k \equiv Q_5$ NS5 leads to the fractional string with the tension $\tilde{T} = T/k$, where T represents the tension of fundamental string. If we measure the radii of T^4 by the unit of string length $l_s = 1/\sqrt{T}$, namely, $R_a = r_a l_s$, the momenta (3.60) becomes

$$p_a = \frac{1}{l_s}\left(\frac{n_a}{r_a} + w^a r_a\right), \qquad \bar{p}_a = \frac{1}{l_s}\left(\frac{n_a}{r_a} - w^a r_a\right). \tag{4.24}$$

For the fractional strings, we should replace the string length l_s with $\tilde{l}_s \equiv \sqrt{k} l_s$. Thus, we obtain

$$p_a = \frac{1}{\sqrt{k}} \times \frac{1}{l_s}\left(\frac{n_a}{r_a} + w^a r_a\right), \qquad \bar{p}_a = \frac{1}{\sqrt{k}} \times \frac{1}{l_s}\left(\frac{n_a}{r_a} - w^a r_a\right), \tag{4.25}$$

and the extra factor $1/\sqrt{k}$ completely compensates the spectrum of missing states.

5. Extension to the case of $H_6 \times T^4/\mathbb{Z}_2$

In this section, we extend our previous analysis to the case of superstring theory on $H_6 \times T^4/\mathbb{Z}_2$ background and the symmetric orbifold $\mathrm{Sym}^M(T^4/\mathbb{Z}_2)$. We again find the good correspondence for the (almost) BPS spectrum.

5.1 Spectrum of superstring on $H_6 \times T^4/\mathbb{Z}_2$

The $\mathbb{Z}_2$-orbifold action acts on the string coordinates as

$$\lambda^{a\dot{a}} \longrightarrow -\lambda^{a\dot{a}}, \qquad Y^{a\dot{a}} \longrightarrow -Y^{a\dot{a}}. \tag{5.1}$$

Moreover, we assume the action on the free bosons H^3 and H^4, which are bosonizations of $\lambda^{a\dot{a}}$, as

$$H_3 \longrightarrow H_3 + \pi, \qquad H_4 \longrightarrow H_4 - \pi, \tag{5.2}$$

so that the space-time supersymmetry is preserved. In fact, we can directly check that all the generators of super pp-wave algebra $\{\mathcal{J}, \mathcal{F}, \mathcal{P}_i, \mathcal{P}_i^*, \mathcal{Q}^{\alpha\beta a}\}$ defined in (3.37) and (3.38) are invariant under (5.1) and (5.2). As for the DDF operators (3.40) and (3.47), the orbifold action reads as

$$\mathcal{A}_n^{a\dot{a}} \longrightarrow -\mathcal{A}_n^{a\dot{a}}, \qquad \mathcal{B}_n^{\alpha\dot{a}} \longrightarrow -\mathcal{B}_n^{\alpha\dot{a}}, \tag{5.3}$$

and the other DDF operators are invariant under this action.

Now, we write down the spectrum of (almost) BPS states.

- **Untwisted sector.** The analysis of the untwisted sector is quite easy. First, we consider the BPS states. All the task we have to do is to leave the $\mathbb{Z}_2$-invariant BPS states in the case of T^4. Only the NS-NS and R-R BPS states are left and the NS-R and R-NS BPS states are projected out. Thus, we obtain 8 BPS states for each p and η, and they are identified with the even cohomology of T^4. As for the almost BPS states, we first consider the following Fock vacua as

$$|i,\bar{i};j,\bar{j};\eta,p;n_a,w^a;(\pm)\rangle = |i,\bar{i};j,\bar{j};\eta,p;n_a,w^a\rangle \pm |i,\bar{i};j,\bar{j};\eta,p;-n_a,-w^a\rangle\,, \quad (5.4)$$

where $|i,\bar{i};j,\bar{j};\eta,p;n_a,w^a\rangle$ represents the physical state with the non-vanishing momenta along T^4, which was defined in (3.61). Clearly, the (NS-NS) BPS states are realized as the special cases of such physical states; that is $|i,\bar{i};0,0;\eta,p;0,0;(+)\rangle$. (R-R BPS states can be obtained by multiplying $\mathcal{B}_0^{+\dot{a}}$ and $\bar{\mathcal{B}}_0^{+\dot{a}}$.) The general physical states are constructed by multiplying the DDF operators and supercharges $\mathcal{Q}^{\pm\mp a}$ as in the case of T^4. We only need the following additional constraint as

$$\begin{aligned} &\sharp\{\mathcal{B}^{\alpha\dot{a}}_{-n},\ \mathcal{A}^{a\dot{a}}_{-m}\} + \sharp\{\bar{\mathcal{B}}^{\alpha\dot{a}}_{-n},\ \bar{\mathcal{A}}^{a\dot{a}}_{-m}\} = \text{even}\,, \quad (\text{for the Fock vacua } |i,\bar{i};\cdots;(+)\rangle)\,,\\ &\sharp\{\mathcal{B}^{\alpha\dot{a}}_{-n},\ \mathcal{A}^{a\dot{a}}_{-m}\} + \sharp\{\bar{\mathcal{B}}^{\alpha\dot{a}}_{-n},\ \bar{\mathcal{A}}^{a\dot{a}}_{-m}\} = \text{odd}\,, \quad (\text{for the Fock vacua } |i,\bar{i};\cdots;(-)\rangle)\,. \end{aligned} \quad (5.5)$$

- **Twisted sectors.** There are 16 twisted sectors that describe stringy excitations around each fixed point of orbifold action. For each twisted sector, we need to consider the following boundary conditions as

$$\begin{aligned} Y^{a\dot{a}}(e^{2\pi i}z) &= -Y^{a\dot{a}}(z)\,, \\ \lambda^{a\dot{a}}(e^{2\pi i}z) &= -\lambda^{a\dot{a}}(z)\,, \qquad (\text{for NS sector})\,, \\ \lambda^{a\dot{a}}(e^{2\pi i}z) &= \lambda^{a\dot{a}}(z)\,, \qquad (\text{for R sector})\,, \end{aligned} \quad (5.6)$$

and moreover,

$$H_3(e^{2\pi i}z) = H_3(z) + \pi\,, \qquad H_4(e^{2\pi i}z) = H_4(z) - \pi\,. \quad (5.7)$$

First, we consider the BPS states. In the NS vacua, there are both of the bosonic and fermionic twist fields; $\sigma^b_{\mathbf{Z}_2}$, $\sigma^f_{\mathbf{Z}_2}$, whose conformal weights are equal to

$$h(\sigma^b_{\mathbf{Z}_2}) = \bar{h}(\sigma^b_{\mathbf{Z}_2}) = 4\times\frac{1}{16} = \frac{1}{4}\,, \qquad h(\sigma^f_{\mathbf{Z}_2}) = \bar{h}(\sigma^f_{\mathbf{Z}_2}) = 4\times\frac{1}{16} = \frac{1}{4}\,. \quad (5.8)$$

Based on this fact we can observe that there are no BPS states in the NS-NS, NS-R and R-NS sectors. On the other hand, the R vacua can include only the bosonic twist field $\sigma^b_{\mathbf{Z}_2}$ and only the spin fields along the H_6 direction, which we express as $S^{\epsilon_0\epsilon_1\epsilon_2}$. This fact leads us to a unique R-R BPS state (per each twisted sector), which is explicitly written as

$$S^{-++}_{-\frac{3}{8}+\eta}\bar{S}^{-++}_{-\frac{3}{8}+\eta}\sigma^b_{\mathbf{Z}_2}|0,\eta,p\rangle\otimes c\bar{c}e^{-\frac{\phi}{2}-\frac{\bar{\phi}}{2}}|0\rangle_{\text{gh}}\,. \quad (5.9)$$

In this way, we have found the 16 R-R BPS states in the twisted sectors for each p and η. They correspond to the blow-up modes of $T^4/\mathbb{Z}_2$ orbifold and reproduce the cohomology ring of $K3$ together with the contributions from the untwisted sector.

The other physical states are also straightforwardly constructed. Contrary to the untwisted sector, the states with non-trivial momenta along T^4 are not allowed. Thus, we only have to consider the states created by the actions of DDF operators over the BPS states (5.9). The only non-trivial point is that the modes of DDF operators $\mathcal{B}_r^{\alpha\dot{a}}$ should be half integers $r \in \frac{1}{2}+\mathbb{Z}$ in this case. This fact originates from the boundary condition (5.7). We again need the constraint

$$\sharp\{\mathcal{B}_{-r}^{\alpha\dot{a}},\ \mathcal{A}_{-m}^{a\dot{a}}\} + \sharp\{\bar{\mathcal{B}}_{-r}^{\alpha\dot{a}},\ \bar{\mathcal{A}}_{-m}^{a\dot{a}}\} = \text{even}, \tag{5.10}$$

to preserve the $\mathbb{Z}_2$-invariance.

5.2 Comparison with $\mathrm{Sym}^M(T^4/\mathbb{Z}_2)$ σ-model

The $\mathbb{Z}_2$-orbifoldization of $\mathrm{Sym}^M(T^4)$ σ-model is defined by the action

$$X_{(A)}^{a\dot{a}} \longrightarrow -X_{(A)}^{a\dot{a}}, \qquad \Psi_{(A)}^{\alpha\dot{a}} \longrightarrow -\Psi_{(A)}^{\alpha\dot{a}}. \tag{5.11}$$

This action preserves the $\mathcal{N}=(4,4)$ superconformal symmetry. We again study the single particle Hilbert space of $\mathbb{Z}_N$-twisted sector with $N = pk+l$ $(0<l<k)$, as in the previous section.

We first consider the spectrum of BPS states. In the untwisted sector of $\mathbb{Z}_2$ orbifoldization, only the $\mathbb{Z}_2$-invariant BPS states survive. They correspond to the even cohomology of T^4.

The analysis of the twisted sectors is more complicated. Focusing on one of the twisted sectors corresponding to the 16 fixed points, we can observe the following aspects:

- $N = 2q+1$

  We have the mode expansions $i\partial X^{a\dot{a}}_{\frac{n}{N}+\frac{1}{2N}}$ and $\Psi^{\alpha\dot{a}}_{\frac{n}{N}}$, $(n\in\mathbb{Z})$ and we obtain for the NS vacuum $|0;N\rangle^{(\mathrm{t})}$

$$\begin{aligned}\hat{L}_0|0;N\rangle^{(\mathrm{t})} &= \left(\frac{N^2-1}{4N}+\frac{1}{2N}\right)|0;N\rangle^{(\mathrm{t})} \equiv \frac{N^2+1}{4N}|0;N\rangle^{(\mathrm{t})}, \\ \hat{K}_0^3|0;N\rangle^{(\mathrm{t})} &= -\frac{1}{2}|0;N\rangle^{(\mathrm{t})}.\end{aligned} \tag{5.12}$$

- $N = 2q$

  We have the mode expansions $i\partial X^{a\dot{a}}_{\frac{n}{N}+\frac{1}{2N}}$ and $\Psi^{\alpha\dot{a}}_{\frac{n}{N}+\frac{1}{2N}}$, $(n\in\mathbb{Z})$ and we obtain for the NS vacuum $|0;N\rangle^{(\mathrm{t})}$

$$\begin{aligned}\hat{L}_0|0;N\rangle^{(\mathrm{t})} &= \left(\frac{N^2-1}{4N}+\frac{1}{4N}\right)|0;N\rangle^{(\mathrm{t})} \equiv \frac{N}{4}|0;N\rangle^{(\mathrm{t})}, \\ \hat{K}_0^3|0;N\rangle^{(\mathrm{t})} &= 0.\end{aligned} \tag{5.13}$$

In these expressions the extra zero-point energies and R-charges assigned to the NS vacua are due to these twisted mode expansions. Based on these aspects, we can find out the following BPS state that is unique for each of the twisted sectors as

$$|\omega^{(1,1)};N\rangle^{(\mathrm{t})} = |\omega^1;N\rangle^{(\mathrm{t})} \otimes \overline{|\omega^1;N\rangle}^{(\mathrm{t})} , \tag{5.14}$$

$$|\omega^1;N\rangle^{(\mathrm{t})} = \prod_{i=0}^{q} \Psi^{++}_{-\frac{i}{N}} \Psi^{+-}_{-\frac{i}{N}} |0;N\rangle^{(\mathrm{t})} , \qquad (N=2q+1) ,$$

$$|\omega^1;N\rangle^{(\mathrm{t})} = \prod_{i=0}^{q-1} \Psi^{++}_{-\frac{i}{N}-\frac{1}{2N}} \Psi^{+-}_{-\frac{i}{N}-\frac{1}{2N}} |0;N\rangle^{(\mathrm{t})} , \qquad (N=2q) , \tag{5.15}$$

and we obtain

$$\hat{L}_0|\omega^1;N\rangle^{(\mathrm{t})} = \hat{K}^3_0|\omega^1;N\rangle^{(\mathrm{t})} = \frac{N}{2}|\omega^1;N\rangle^{(\mathrm{t})} . \tag{5.16}$$

In summary, we have obtained $8+16=24$ BPS states, which precisely correspond to the cohomology of $K3$ for each of the $\mathbb{Z}_N$-twisted sectors. They have (approximately) degenerate charges $\mathcal{F}(\equiv (\hat{K}^3_0+\hat{L}_0)/k) = p + l/k$, and $\mathcal{J}(\equiv \hat{K}^3_0 - \hat{L}_0) = 0$. As in the case of T^4, we have a good correspondence between the string theory and symmetric orbifold theory under the identifications $N = kp + l$ and $\eta = l/k$.

With respect to the almost BPS states, the discussion is almost parallel to the case of T^4. We can explicitly write down the complete list of almost BPS states in the symmetric orbifold theory and compare it with the string theory result by using the identification of DDF operators, as before. However, in this case, we must identify the DDF operators $\mathcal{B}^{\alpha\dot{a}}_{n+1/2}$ (where $n \in \mathbb{Z}$) in the twisted sectors with $\Psi^{\pm\dot{a}}_{\mp\frac{1}{2}+\frac{nk}{N}+\frac{k}{2N}}$ rather than $\Psi^{\pm\dot{a}}_{\mp\frac{1}{2}+\frac{nk}{N}}$. (We again neglect the small difference of mode $1/2N$.) The short string spectrum is again completely embedded in the Hilbert space of symmetric orbifold theory and there are many missing states.

6. Conclusion

In this paper we have studied the correspondence between string theory on pp-wave background with NSNS-flux and superconformal theory on symmetric orbifold $\mathrm{Sym}^{Q_1Q_5}(M^4)$, where $M^4 = T^4$ or $T^4/\mathbb{Z}_2(\cong K3)$. Superstring theory on the pp-wave background is obtained as the Penrose limit of superstring on the $AdS_3 \times S^3 \times M^4$ with NSNS-flux [27]. This theory is described by a noncompact WZW model with the target manifold of 6-dimensional Heisenberg group (H_6). We employed current algebra approach according to [28] and quantize the system in the covariant gauge. By making use of the free field representation, we have explicitly constructed the physical vertices that correspond to the generators of global supersymmetries and the complete set of DDF operators. The spectrum of physical states is classified by the “short string sectors” and “long string sectors”, as in the AdS_3 string theory [25]. The latter has a continuous spectrum of light-cone energy and the strings freely propagate along the transverse plane, while the strings of the former cannot propagate along the transverse plane.

We have compared the general short string excitations with the single particle Hilbert spaces included in the symmetric orbifold theory. We have shown that all the the short string states are successfully embedded into the Hilbert space of symmetric orbifold theory. At this point, our analysis of DDF operators played an essential role. We have also found the existence of many missing states in the string Hilbert space, which should be understood as non-perturbative excitations.

It has just begun to study the duality between string theory on pp-wave backgrounds and boundary conformal field theory. Among other things, it is a very interesting and challenging problem how we should explain the origin of missing states mentioned above. For this purpose it may be a suggestive thing that our H_6 string theory can be described by the conformal theory reminiscent of $\mathbb{C}^2/\mathbb{Z}_k$ model under our identification $\eta = l/k$. Roughly speaking, this seems to originate from the well-known fact that the k NS5 system should be the T-dual of $\mathbb{C}^2/\mathbb{Z}_k \sim ALE(A_{k-1})$, although the original $AdS_3 \times S^3$ background is known to be located at the different point in the moduli space.[7] In any case, the similarity with the $\mathbb{C}^2/\mathbb{Z}_k$ model seems to imply the existence of fractional string excitations, if we take a non-perturbative approach, say, the matrix string theory [38]. As we discussed in section 4, such excitations may give a good explanation of the missing states.

Another important problem is to determine the interactions from the world-sheet techniques by calculating the correlation functions. Recently, the interactions in string theory on pp-wave backgrounds with RR-flux have been investigated [12]. It may be interesting to analyse the string interaction in our NSNS-model and compare it with the results of the RR-background.

Acknowledgments

We would like to thank T. Takayanagi for useful discussions. The work of Y. S. is supported in part by a Grant-in-Aid for the Encouragement of Young Scientists (♯13740144) from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

A. Gamma matrices

The cocycle factors of spin fields are defined by means of the gamma matrices. We here summarize the convention in this paper;

$$\begin{aligned}
\Gamma_{\pm 0} &= \sigma_{\pm} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\,, \\
\Gamma_{\pm 1} &= \sigma_3 \otimes \sigma_{\pm} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}\,, \\
\Gamma_{\pm 2} &= \sigma_3 \otimes \sigma_3 \otimes \sigma_{\pm} \otimes \mathbf{1} \otimes \mathbf{1}\,, \\
\Gamma_{\pm 3} &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_{\pm} \otimes \mathbf{1}\,, \\
\Gamma_{\pm 4} &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_{\pm}\,,
\end{aligned} \tag{A.1}$$

[7] The $\mathbb{C}^2/\mathbb{Z}_k$ model corresponds to the point with non-vanishing world-sheet theta angle $\theta = 1/k$, but the $AdS_3 \times S^3$ model should correspond to the point of $\theta = 0$. However, the difference become negligible in the large k limit.

where we use the usual Pauli matrices as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.2}$$

and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. We use the charge conjugation matrix as

$$C = \epsilon \otimes \sigma_1 \otimes \epsilon \otimes \sigma_1 \otimes \epsilon \,, \qquad \epsilon = i\sigma_2 \,, \tag{A.3}$$

which has the property as

$$C\Gamma_\mu C^{-1} = -(\Gamma_\mu)^T \,, \qquad C^\dagger = C^{-1} \,. \tag{A.4}$$

Then the OPEs including spin fields are given as follows

$$\begin{aligned} \psi^\mu(z) S^A(w) &\sim \frac{1}{(z-w)^{\frac{1}{2}}} (\Gamma^\mu)^A_B S^B(w) \,, \\ \psi^\mu \psi^\nu(z) S^A(w) &\sim -\frac{1}{z-w} (\Gamma^{\mu\nu})^A_B S^B(w) \,, \\ S^A(z) S^B(w) &\sim \frac{1}{(z-w)^{\frac{3}{4}}} (\Gamma_\mu C)^{AB} \psi^\mu(w) \,. \end{aligned} \tag{A.5}$$

B. Supersymmetry of DDF operators

Supertransformations of DDF operators under the action of $\mathcal{Q}^{\pm\mp a}$ are given in (3.43), (3.44) and (3.49). In this appendix, we write them in explicit forms. For DDF operators corresponding to "affine" H_6 extension, they are given by

$$\begin{aligned} [\mathcal{Q}^{-+a}, \mathcal{P}_{1,n}] &= \frac{n+\eta}{p+\eta} \mathcal{Q}^{++a}_n \,, & [\mathcal{Q}^{+-a}, \mathcal{P}^*_{1,n}] &= \frac{n-\eta}{p+\eta} \mathcal{Q}^{--a}_n \,, \\ [\mathcal{Q}^{+-a}, \mathcal{P}_{2,n}] &= \frac{n+\eta}{p+\eta} \mathcal{Q}^{++a}_n \,, & [\mathcal{Q}^{-+a}, \mathcal{P}^*_{2,n}] &= -\frac{n-\eta}{p+\eta} \mathcal{Q}^{--a}_n \,, \\ \{\mathcal{Q}^{+-a}, \mathcal{Q}^{++b}_n\} &= -\epsilon^{ab} \mathcal{P}_{1,n} \,, & \{\mathcal{Q}^{-+a}, \mathcal{Q}^{--b}_n\} &= \epsilon^{ab} \mathcal{P}^*_{1,n} \,, \\ \{\mathcal{Q}^{-+a}, \mathcal{Q}^{++b}_n\} &= \epsilon^{ab} \mathcal{P}_{2,n} \,, & \{\mathcal{Q}^{+-a}, \mathcal{Q}^{--b}_n\} &= \epsilon^{ab} \mathcal{P}^*_{2,n} \,. \end{aligned} \tag{B.1}$$

For DDF operators of T^4 sectors, they are given by

$$\begin{aligned} [\mathcal{Q}^{+-a}, \mathcal{A}^{b\dot b}_n] &= \frac{n}{p+\eta} \epsilon^{ab} \mathcal{B}^{-\dot b}_n \,, & [\mathcal{Q}^{-+a}, \mathcal{A}^{b\dot b}_n] &= -\frac{n}{p+\eta} \epsilon^{ab} \mathcal{B}^{+\dot b}_n \,, \\ \{\mathcal{Q}^{-+a}, \mathcal{B}^{-\dot b}_n\} &= -\mathcal{A}^{a\dot b}_n \,, & \{\mathcal{Q}^{+-a}, \mathcal{B}^{+\dot b}_n\} &= -\mathcal{A}^{a\dot b}_n \,. \end{aligned} \tag{B.2}$$

References

[1] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242]; *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081];
P. Meessen, *A small note on pp-wave vacua in 6 and 5 dimensions*, *Phys. Rev.* **D 65** (2002) 087501 [hep-th/0111031];

M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, hep-th/0202111;
M. Cvetič, H. Lu and C.N. Pope, *Penrose limits, pp-waves and deformed M2-branes*, hep-th/0203082; *M-theory pp-waves, Penrose limits and supernumerary supersymmetries*, hep-th/0203229;
J.P. Gauntlett and C.M. Hull, *pp-waves in 11-dimensions with extra supersymmetry*, hep-th/0203255.

[2] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044];
R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[3] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[5] N. Itzhaki, I.R. Klebanov and S. Mukhi, *pp wave limit and enhanced supersymmetry in gauge theories*, *J. High Energy Phys.* **03** (2002) 048 [hep-th/0202153];
J. Gomis and H. Ooguri, *Penrose limit of $N = 1$ gauge theories*, hep-th/0202157;
L.A. Pando Zayas and J. Sonnenschein, *On Penrose limits and gauge theories*, *J. High Energy Phys.* **05** (2002) 010 [hep-th/0202186];
M. Alishahiha and M.M. Sheikh-Jabbari, *The pp-wave limits of orbifolded $AdS_5 \times S^5$*, *Phys. Lett.* **B 535** (2002) 328 [hep-th/0203018];
N.-W. Kim, A. Pankiewicz, S.-J. Rey and S. Theisen, *Superstring on pp-wave orbifold from large-N quiver gauge theory*, hep-th/0203080;
T. Takayanagi and S. Terashima, *Strings on orbifolded pp-waves*, hep-th/0203093;
E. Floratos and A. Kehagias, *Penrose limits of orbifolds and orientifolds*, hep-th/0203134;
S. Mukhi, M. Rangamani and E. Verlinde, *Strings from quivers, membranes from moose*, *J. High Energy Phys.* **05** (2002) 023 [hep-th/0204147];
M. Alishahiha and M.M. Sheikh-Jabbari, *Strings in pp-waves and worldsheet deconstruction*, hep-th/0204174;
K. Oh and R. Tatar, *Orbifolds, Penrose limits and supersymmetry enhancement*, hep-th/0205067;
S.D. Mathur, A. Saxena and Y.K. Srivastava, *Scalar propagator in the pp-wave geometry obtained from $AdS_5 \times S^5$*, hep-th/0205136.

[6] M. Billo and I. Pesando, *Boundary states for GS superstrings in an Hpp wave background*, *Phys. Lett.* **B 536** (2002) 121 [hep-th/0203028];
C.-S. Chu and P.-M. Ho, *Noncommutative D-brane and open string in pp-wave background with B-field*, hep-th/0203186;
A. Dabholkar and S. Parvizi, *Dp-branes in pp-wave background*, hep-th/0203231;
D. Berenstein, E. Gava, J.M. Maldacena, K.S. Narain and H. Nastase, *Open strings on plane waves and their Yang-Mills duals*, hep-th/0203249;
P. Lee and J.-W. Park, *Open strings in pp-wave background from defect conformal field theory*, hep-th/0203257;
A. Kumar, R.R. Nayak and Sanjay, *D-brane solutions in pp-wave background*, hep-th/0204025;
D.-S. Bak, *Supersymmetric branes in pp wave background*, hep-th/0204033;
K. Skenderis and M. Taylor, *Branes in AdS and pp-wave spacetimes*, hep-th/0204054;

V. Balasubramanian, M.-X. Huang, T.S. Levi and A. Naqvi, *Open strings from $N = 4$ super Yang-Mills*, hep-th/0204196;
H. Takayanagi and T. Takayanagi, *Open strings in exactly solvable model of curved space-time and pp-wave limit*, *J. High Energy Phys.* **05** (2002) 012 [hep-th/0204234];
P. Bain, P. Meessen and M. Zamaklar, *Supergravity solutions for D-branes in Hpp-wave backgrounds*, hep-th/0205106;
M. Alishahiha and A. Kumar, *D-brane solutions from new isometries of pp-waves*, hep-th/0205134.

[7] J.G. Russo and A.A. Tseytlin, *On solvable models of type-IIB superstring in NS-NS and R-R plane wave backgrounds*, *J. High Energy Phys.* **04** (2002) 021 [hep-th/0202179].

[8] M. Hatsuda, K. Kamimura and M. Sakaguchi, *From super-$AdS_5 \times S^5$ algebra to super-pp-wave algebra*, *Nucl. Phys.* **B 632** (2002) 114 [hep-th/0202190]; *Super-pp-wave algebra from super-$AdS \times S$ algebras in eleven-dimensions*, hep-th/0204002.

[9] U. Gursoy, C. Nunez and M. Schvellinger, *RG flows from* Spin(7)*, CY 4-fold and HK manifolds to AdS, Penrose limits and pp waves*, hep-th/0203124;
J. Michelson, *(Twisted) toroidal compactification of pp-waves*, hep-th/0203140;
R. Gueven, *Randall-Sundrum zero mode as a Penrose limit*, *Phys. Lett.* **B 535** (2002) 309 [hep-th/0203153];
S.R. Das, C. Gomez and S.-J. Rey, *Penrose limit, spontaneous symmetry breaking and holography in pp-wave background*, hep-th/0203164;
H. Lu and J.F. Vazquez-Poritz, *Penrose limits of non-standard brane intersections*, hep-th/0204001;
R.G. Leigh, K. Okuyama and M. Rozali, *pp-waves and holography*, hep-th/0204026;
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, hep-th/0204051;
Y. Imamura, *Large angular momentum closed strings colliding with D-branes*, hep-th/0204200;
S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 007 [hep-th/0204226];
I. Bakas and K. Sfetsos, *pp-waves and logarithmic conformal field theories*, hep-th/0205006;
C.-H. Ahn, *More on Penrose limit of $AdS_4 \times Q^{1,1,1}$*, hep-th/0205008; *Comments on Penrose limit of $AdS_4 \times M^{1,1,1}$*, hep-th/0205109;
H. Singh, *M5-branes with 3/8 supersymmetry in pp-wave background*, hep-th/0205020;
M. Li, *Correspondence principle in a pp-wave background*, *J. High Energy Phys.* **05** (2002) 033 [hep-th/0205043].

[10] N. Berkovits, *Conformal field theory for the superstring in a Ramond-Ramond plane wave background*, *J. High Energy Phys.* **04** (2002) 037 [hep-th/0203248].

[11] E. Kiritsis and B. Pioline, *Strings in homogeneous gravitational waves and null holography*, hep-th/0204004.

[12] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background*, hep-th/0204146;
C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and pp-wave strings*, hep-th/0205033;
D. Berenstein and H. Nastase, *On lightcone string field theory from super Yang-Mills and holography*, hep-th/0205048;
D.J. Gross, A. Mikhailov and R. Roiban, *Operators with large R charge in $N = 4$ Yang-Mills theory*, hep-th/0205066;

N.R. Constable et al., *pp-wave string interactions from perturbative Yang-Mills theory*, hep-th/0205089;
R. Gopakumar, *String interactions in pp-waves*, hep-th/0205174.

[13] A. Parnachev and D.A. Sahakyan, *Penrose limit and string quantization in $AdS_3 \times S^3$*, hep-th/0205015.

[14] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, *Phys. Rev. Lett.* **71** (1993) 3751 [hep-th/9310112].

[15] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS_3*, *Adv. Theor. Math. Phys.* **2** (1998) 733 [hep-th/9806194].

[16] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[17] J.M. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *J. High Energy Phys.* **12** (1998) 005 [hep-th/9804085].

[18] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, *String theory on AdS_3*, *J. High Energy Phys.* **12** (1998) 026 [hep-th/9812046].

[19] D. Kutasov and N. Seiberg, *More comments on string theory on AdS_3*, *J. High Energy Phys.* **04** (1999) 008 [hep-th/9903219].

[20] D. Kutasov, F. Larsen and R.G. Leigh, *String theory in magnetic monopole backgrounds*, *Nucl. Phys.* **B 550** (1999) 183 [hep-th/9812027].

[21] K. Hosomichi and Y. Sugawara, *Hilbert space of space-time SCFT in AdS_3 superstring and T^{4kp}/S_{kp} SCFT*, *J. High Energy Phys.* **01** (1999) 013 [hep-th/9812100].

[22] Y. Hikida, K. Hosomichi and Y. Sugawara, *String theory on AdS_3 as discrete light-cone Liouville theory*, *Nucl. Phys.* **B 589** (2000) 134 [hep-th/0005065].

[23] R. Argurio, A. Giveon and A. Shomer, *Superstrings on AdS_3 and symmetric products*, *J. High Energy Phys.* **12** (2000) 003 [hep-th/0009242].

[24] M. Henningson, S. Hwang, P. Roberts and B. Sundborg, *Modular invariance of SU(1,1) strings*, *Phys. Lett.* **B 267** (1991) 350.

[25] J.M. Maldacena and H. Ooguri, *Strings in AdS_3 and $\mathrm{SL}(2,\mathbb{R})$ WZW model, 1*, *J. Math. Phys.* **42** (2001) 2929 [hep-th/0001053].

[26] R.R. Metsaev and A.A. Tseytlin, *Superparticle and superstring in $AdS_3 \times S^3$ Ramond-Ramond background in light-cone gauge*, *J. Math. Phys.* **42** (2001) 2987 [hep-th/0011191].

[27] K. Sfetsos, *Gauging a nonsemisimple WZW model*, *Phys. Lett.* **B 324** (1994) 335 [hep-th/9311010];
D.I. Olive, E. Rabinovici and A. Schwimmer, *A class of string backgrounds as a semiclassical limit of WZW models*, *Phys. Lett.* **B 321** (1994) 361 [hep-th/9311081];
K. Sfetsos and A.A. Tseytlin, *Four-dimensional plane wave string solutions with coset CFT description*, *Nucl. Phys.* **B 427** (1994) 245 [hep-th/9404063].

[28] E. Kiritsis and C. Kounnas, *String propagation in gravitational wave backgrounds*, *Phys. Lett.* **B 320** (1994) 264 [hep-th/9310202];
E. Kiritsis, C. Kounnas and D. Lust, *Superstring gravitational wave backgrounds with space-time supersymmetry*, *Phys. Lett.* **B 331** (1994) 321 [hep-th/9404114].

[29] O. Jofre and C. Nunez, *Strings in plane wave backgrounds revisited*, *Phys. Rev.* **D 50** (1994) 5232 [hep-th/9311187];
N. Mohammedi, *On bosonic and supersymmetric current algebras for nonsemisimple groups*, *Phys. Lett.* **B 325** (1994) 371 [hep-th/9312182];
A. Kumar and S. Mahapatra, *Exact duality and nilpotent gauging*, *Mod. Phys. Lett.* **A 9** (1994) 925 [hep-th/9401098];
J.M. Figueroa-O'Farrill and S. Stanciu, *Nonsemisimple Sugawara constructions*, *Phys. Lett.* **B 327** (1994) 40 [hep-th/9402035];
A.A. Kehagias and P.A.A. Meessen, *Exact string background from a WZW model based on the Heisenberg group*, *Phys. Lett.* **B 331** (1994) 77 [hep-th/9403041];
I. Antoniadis and N.A. Obers, *Plane gravitational waves in string theory*, *Nucl. Phys.* **B 423** (1994) 639 [hep-th/9403191];
P. Forgács, P.A. Horváthy, Z. Horváth and L. Palla, *The Nappi-Witten string in the light-cone gauge*, *Heavy Ion Phys.* **1** (1995) 65 [hep-th/9503222].

[30] J.G. Russo and A.A. Tseytlin, *Constant magnetic field in closed string theory: an exactly solvable model*, *Nucl. Phys.* **B 448** (1995) 293 [hep-th/9411099]; *Exactly solvable string models of curved space-time backgrounds*, *Nucl. Phys.* **B 449** (1995) 91 [hep-th/9502038]; *Heterotic strings in uniform magnetic field*, *Nucl. Phys.* **B 454** (1995) 164 [hep-th/9506071];
A.A. Tseytlin, *Closed superstrings in magnetic flux tube background*, *Nucl. Phys.* **49** *(Proc. Suppl.)* (1996) 338 [hep-th/9510041].

[31] J.M. Maldacena, J. Michelson and A. Strominger, *Anti-de Sitter fragmentation*, *J. High Energy Phys.* **02** (1999) 011 [hep-th/9812073].

[32] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *J. High Energy Phys.* **04** (1999) 017 [hep-th/9903224].

[33] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678; *Strings on orbifolds, 2*, *Nucl. Phys.* **B 274** (1986) 285;
L.J. Dixon, D. Friedan, E. Martinec and H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.

[34] M. Fukuma, H. Kawai and R. Nakayama, *Infinite dimensional Grassmannian structure of two-dimensional quantum gravity*, *Commun. Math. Phys.* **143** (1992) 371.

[35] L. Borisov, M.B. Halpern and C. Schweigert, *Systematic approach to cyclic orbifolds*, *Int. J. Mod. Phys.* **A 13** (1998) 125 [hep-th/9701061].

[36] Y. Sugawara, *$N = (0,4)$ quiver SCFT$_2$ and supergravity on $AdS_3 \times S^2$*, *J. High Energy Phys.* **06** (1999) 035 [hep-th/9903120].

[37] K. Hosomichi and Y. Sugawara, *Multi-strings on $AdS_3 \times S^3$ from matrix string theory*, *J. High Energy Phys.* **07** (1999) 027 [hep-th/9905004].

[38] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030];
L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025;
T. Banks and N. Seiberg, *Strings from matrices*, *Nucl. Phys.* **B 497** (1997) 41 [hep-th/9702187].